GR.

SCIAMA

Plan
i) Physical motivation – inertia (1)
ii) Differential geometry (5)
iii) Gravitational field eqns., solutions, etc. (10)

Physical ideas in a sense Newtonian. No new expts. (unlike S. Rel.!)
Like Newtonian, S. Rel. assumes ∃ inertial frames without question. In such frames Newton's laws hold. S. Rel. differs from N. only by transformation between frames.
Thus a sort of absolute space still exists, ∵ only if obs. is unaccelerated "rel. to it" is he in an inertial frame.
Note: no connexion with gravitation at this pt. It is implication of inertial frames that matters.
Eqn of motion in inertial frame of a free particle $\frac{d^2x^i}{ds^2} = 0$. s is proper time

In non-inertial frame, $\frac{d^2x^i}{ds^2}$ + inertial forces (centrifugal, Coriolis, etc) = 0
introduced to get correct dynamics.

Mathematically, this is ≡t. to saying that geodesic eqn. is particularly simple in a co-ordinate system of an inertial frame. In a non-inertial frame, get curvilinear co-ordinates.
Newton — inertial forces "fictitious" (result of our clumsiness in frame choice)
Now – forces produced by matter.
Berkeley. Mach.
Mach's Principle. (broadly) Inertial forces are not fictitious, but are produced by matter.
If measure by local expts. which frame is most nearly inertial (i.e. N's laws hold), then we find that inertial frames are unaccelerated w.r.t. stars. Foucault pendulum, etc. Observation.
B + M suppose that this is not a coincidence.
Hence i) stars (galaxies, etc.) constitute bulk of matter in Universe & are source of inertial force.
ii) when stars are accelerated rel. to you, they exert these forces.
Also distant matter must be most important, as we do not detect influence of local lumps.

Nature of the interaction? i) new force tailored for the job. Better than absolute space?
ii) Einstein. Principle of Equivalence (1907).
Inertial forces ∝ inertial mass of object. (∴ electrical forces could not cause inertia)
Expt. also shows that gravl. forces ∝ inertial mass. Dicke recently, Newton then.
∴ Does gravitation → Mach's Principle?
Einstein. Inertial forces are gravitational forces, exerted by stars when they accelerate.
Physicist. Gravitation is $1/r^2$, indepᵗ (nearly, to take expt. seriously) of motion.
Problem: how to link $1/r^2$ & acceleration-dependence.
Hope – Accelerating charge produces forces other than Coulomb. (long-range also)
∴ nothing yet to worry about. Demands are not ridiculous.
Mathematician (Einstein in particular)
Modification to geodesic eqn. ≡t. to geometrical properties of space-time.
Keep idea that particle follows geodesic but that s-t is curved. Inertial forces become geometrical properties of space-time. Basic idea: try the hypothesis that particle moving under gravl. forces + inertial forces still moves on a geodesic, now not of Minkowskian type, but curved.

In curved s-t. extra terms will give inertial forces + the inverse-square force

Mink. " " " " " " - only.

Characteristic of curved s-t. that term cannot be made to go $\to 0$.

$\therefore$ cannot get inertial frame — property of $1/r^2$ grav. force.

Non-rigorous Differential Geometry.

Will not consider whether or not we are restricted to neighbourhoods of pts. Topology reqd. for global problem. Can whole space be covered by non-singular co-ordinate system? e.g. spherical polars singular at $\theta = 0$, ϕ - dir.ⁿ not defined. Assume that co-ordinate systems covering local spaces can be transformed regularly to those covering nearby spaces. (Diff.ˡ geometry in "the small".) Also assume all fns. as regular as we need.

Begin with a featureless space - "unconnected manifold". No connexion in props. bet. neighbouring pts. Call it 4-d for later. Could be n-d for generality, but will look ahead. Space is a 4-d continuum of distinguishable pts. We label these pts. with 4 cts. labels x^1, x^2, x^3, x^4. We permit co-ord transformations $x'^1 = x'^1(x^1, x^2, x^3, x^4)$

$$x'^2 = x'^2(x^1, x^2, x^3, x^4)$$

will be interested in case where fns. are as regular as we want, differentiable, etc. Non-zero fnl. det. so that they are invertible.

Scalar qty. ϕ at a pt. is unaltered by co-ord transf.ⁿ

Scalar field ϕ defined at each pt. of a space.

Vectors. i) Contravariant Vector. A^k Defined to transform like dx^k, small co-ord. diffce.

$$\text{i.e.} \quad dx'^k = \frac{\partial x'^k}{\partial x^l} dx^l \qquad (sc)$$

Note that this transf.ⁿ is transitive, second transf $\to$ (1st $\times$ 2nd).

$$A'^k = \frac{\partial x'^k}{\partial x^l} A^l$$

ii) Covariant Vector. B_k Transforms so that $A^i B_i$ is a scalar.

$$B'_k = \frac{\partial x^l}{\partial x'^k} B_l$$

$$\text{e.g.} \quad \frac{\partial \phi}{\partial x^k}$$

Suppose $\exists$ x^i, y^i, two pts. in 1 co-ord syst. $\phi(y^i) - \phi(x^i)$ is a scalar $\because \phi$ scalar

As $y^i \to x^i$, $\to \dfrac{\partial \phi}{\partial x^i} dx^i$, also a scalar.

But as dx^i is contravariant, $\partial\phi/\partial x^i$ must be cov.ᵗ

Geometrically, dx^i displ.ᵗ in surf., cov. is a "normal"

contrav.

At a glance: Upstairs index in denom $\equiv$ Downstairs index.

(For orthog.l transf.ns in Euclid. sp., cov & con transform in same way.)

To get tensor transf.n law. Consider as outer product of vectors.

i.e. $A_i B_j \ldots\ldots G^k H^l \ldots$

Only expressible tho' as sum of suff.t no. of such prods.

Then contrav. as regards k, l indices, cov. as regards i, j . . . indices.

Comments.

i) Contraction. $A^{klm\ldots\ldots}{}_{pq\ldots\ldots}$ consider $k \equiv p$. $\rightarrow A^{klm\ldots}{}_{kq\ldots}$

k's then no longer influence transf. props. $\rightarrow A^{lm}{}_{q}$

Only always a tensor if 1 upstairs & 1 d'stairs. put =.r

Will only be interested throughout in operations that turn tensors into tensors.

This ∴ tensor laws are invariant, hold in all co-ord systs. if hold in one. ($S^{kl} = T^{kl}$)

Note that these tensors in a phys. law must hold at same pt.

or tensor field must be def. on same spaces.

ii) Sym + skew-sym. $S^{kl\ldots}{}_{pq\ldots} = \pm S^{lk\ldots}{}_{pq\ldots}$

$+ \rightarrow$ sym. in k, l

$- \rightarrow$ asym in k, l.

Both indices u'stairs or d'stairs for this, for result still to be tensor.

Not.n: $T_{kl} = A_{\underline{kl}} + B_{\underset{\vee}{kl}}$ (or $A_{kl} = A_{\underline{kl}} + A_{\underset{\vee}{kl}}$)

$A_{\underline{kl}} = \frac{1}{2}(A_{kl} + A_{lk})$

$A_{\underset{\vee}{kl}} = \frac{1}{2}(A_{kl} - A_{lk})$

iii) To tell whether qty. tensor or not.

Ⓐ. If $S^{\cdots}{}_{\cdots\cdots}$ o.r. $S^{\cdots}{}_{\cdots\cdots}$ A. B. F' G' has no indices, is scalar for all choices of A., B., F', G' etc., then S is a tensor.

Π. Let S' be transf.n of S via tensor law.

& S'' be another set of nos. sat.g conds. of Ⓐ.

Then $(S'^{kl\ldots}{}_{pq\ldots} - S''^{kl\ldots}{}_{pq\ldots}) A'_k B'_l F'^p G'^q = 0$.

Choose A, B etc. o.r. of A' only the kth, B' only the lth cpts, etc. $\neq 0$.

Then $(S'^{kl}{}_{pq\ldots} - S''^{kl}{}_{pq\ldots}) = 0$ k, l, p, q. now have part.r vals.

Do this now for all cpts. By choice of A, B etc. suitably. Then $S' \equiv S''$.

Similarly, if $S^{kl} B_l = A^k$ for all A, then S is a tensor.

if $S^{kl} A_k A_l = \phi$, $S^{\underline{kl}}$ only need be tensor.

iv) "Unit tensor" δ^i_k (like Kronecker δ) (note pos.n of indices)

$\delta^i_k A_i B^k = A_k B^k$ = scalar for all A, B. ∴ δ^i_k is a tensor.

But it has special property of numerical invariance.

v) Tensor densities. Also → invariant relationships.

? Integrals. Integral is sum of qtys. defined at diffr. pts. in space

$A_k + B_k = C_k$ makes sense (for us) only if A & B are def. on same pt.
Algebra of tensors & vectors must be def. for pt. (except for scalars).

Suppose we were interested in $\int A\, dx^1 dx^2 dx^3 dx^4 = \int A\, d\tau$.
Reasonable to require that it is a scalar. $\int A\, d\tau = \int A' d\tau'$
We know how $d\tau$ transforms, like change of variable in an $\int$al.
Therefore our reqt. of scalarity requires further

$$A' = \left|\frac{\partial x^k}{\partial x'^i}\right| A$$

A is then what is meant by a scalar density. "Scalar" ∵ transfmn. is not vector or tensor, but × a no. "Density" ∵ vol. $\int$al is a scalar. A commonly printed as $\mathcal{A}$

Sim., tensor density is qty transf. like tensor, but in addⁿ takes on factor of det. of transf.
No question of integrating one of these, name "density" carried over from scalar density.
Eqns. for tensor densities will also be invariant.

Slightly diffᵗ sitⁿ arises if transf. takes on some power of determinant other than unity.
"Densities of different weights". Weight ≡ power to which determinant raised in transfmⁿ.

Exx. of scalar densities.

Consider T_{klmn}, skew-sym in all pairs of indices.
Let J be particular cpt. T_{1234}
What is T'_{1234}? , $= \frac{\partial x_k}{\partial x'_1}\cdot\frac{\partial x_l}{\partial x'_2}\cdot\frac{\partial x_m}{\partial x'_3}\cdot\frac{\partial x_n}{\partial x'_4} T_{klmn}$ (indices should be ursts!)

Now ∵ of skew-sym. $T_{klmn} = \pm J$ acc. as. $klmn$ is even or odd permⁿ of 1234.
∴ from defⁿ of a determinant, $T'_{1234} = \left|\frac{\partial x^k}{\partial x'^i}\right| J$

$$\therefore J' = T'_{1234} = \left|\frac{\partial x^k}{\partial x'^i}\right| J$$

∴ J is a scalar density.

This can be reversed. Start with scalar A. Then define $\mathcal{A}^{klmn} = \pm A$ as $klmn$ is even or odd.
Then $\mathcal{A}^{klmn}$ is completely skew tensor density of rank 4.
If $A=1$, the qty $\mathcal{A}$ is ε^{klmn}, familiar from Cartesian tensors, numerically inv. like δ^i_k.

In Cartesian systems, the transfⁿ determinant always $=1$. ∴ everything simpler.

$\delta_{ik} \longrightarrow$	δ^i_k	$\varepsilon_{klmn} \rightarrow$	ε^{klmn}
Cartesⁿ	Tensor	Cartesⁿ	Tensor <u>Density</u>.

<u>Formⁿ of scalar density from any covariant</u> g_{ij}

$$g'_{ik} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k} g_{lm}$$

1 3 2

Regard this as matrix prod. of 1, 2, 3.
Then $\det(g'_{ik}) \equiv g' = \left|\frac{\partial x^i}{\partial x'^j}\right|^2 g$ $\qquad \sqrt{g'} = \left|\frac{\partial x^i}{\partial x'^j}\right|\sqrt{g}$

$\sqrt{g}$ is a scalar density of weight 1.

$\sqrt{\ }$ of der. of <u>any</u> 2nd-rank cov. tensor is a scalar density

Transform of cofactors.

Let cof. of g_{ik} be M^{ik} in g. (Put indices there in anticip.^{n}!)
Then

$$g_{mk} M^{lk} = g\,\delta^l_m$$

$$\frac{g_{mk} M^{lk}}{g} = \delta^l_m$$

$\therefore$ M^{lk}/g essentially inverse matrix to g_{mk}.

Define M^{lk} to be cof. of g_{lk} in all frames of refce. Then $g_{mk}\frac{M^{lk}}{g} = \delta^l_m$ holds in all frames. This $=^n$ defines a unique inverse if $g \neq 0$. If it holds in one frame it holds in another if M^{lk}/g is a tensor, g_{mk} & δ^l_m being tensors.
$\therefore$ M^{lk}/g is a tensor $\because$ there is only 1 / frame when $g \neq 0$.

$\therefore$ Normalised cofactors of any 2nd-rank cov. tensor $\rightarrow$ contrav. tensor.
" " " " " " contrav. " cov. " By similarity.

All this pretty obvious – algebra of tensors. What about tensor analysis?

Differentiation

Must obv. talk of tensor fields, defined at all pts.
Different^n means subtraction of tensors at two differ. pts., normalised by interval.
But cannot in general subtract tensors at differ. pts & still get a tensor.

Found that $\frac{\partial\varphi}{\partial x_i}$ was a cov. vector. φ same at all transf^{ns}
Try $\partial/\partial x^k\left(\frac{\partial\varphi}{\partial x^l}\right) = \frac{\partial^2\varphi}{\partial x^k \partial x^l}$ Then $\frac{\partial^2\varphi}{\partial x'^k \partial x'^l} = \frac{\partial x^l}{\partial x'^k}\cdot\frac{\partial x^m}{\partial x'^i}\frac{\partial^2\varphi}{\partial x^l \partial x^m} + \frac{\partial^2 x^l}{\partial x'^k \partial x'^i}\frac{\partial\varphi}{\partial x^l}$

var.^{n} of transf. from place to place. Causes non-tensor character.

$\therefore$ $\partial^2\varphi/\partial x^k \partial x^l$ not a tensor.
$= 0$ in one frame $\not\Rightarrow$ $= 0$ in another.

Similarly, $\frac{\partial A'_k}{\partial x^i} = \frac{\partial x^l}{\partial x'^k}\cdot\frac{\partial x^m}{\partial x'^i}\frac{\partial A_l}{\partial x^m} + \frac{\partial^2 x^l}{\partial x'^i \partial x'^k} A_l$

$\therefore$ $\frac{\partial A_l}{\partial x^m} = A_{l,m}$ not a tensor. $\longleftarrow$ Not.^{n}

Must impose some structure to be able to differentiate.

Affinely Connected Manifold.
Two points of view.

i) Want to be able to construct derivative of a vector field, $\because$ we'll want to say that fields are "constant" & mean something by it

Suppose then that $A_{k,l} = 0$ for a particular co-ord syst.
In 2nd system, $\rightarrow$ $\frac{\partial A'_k}{\partial x'^l} - \frac{\partial^2 x^l}{\partial x'^i \partial x'^k} A_l = 0$

or $\dfrac{\partial A'_k}{\partial x'^i} - \dfrac{\partial x'^n}{\partial x^l}\dfrac{\partial^2 x^l}{\partial x'^i \partial x'^k} A'_n = 0$

$$\downarrow$$

$$\Gamma'^n_{ki} \quad \underline{\text{Not}^n}$$

$\dfrac{\partial A'_k}{\partial x'^i} - \Gamma'^n_{ki} A'_n = 0$ expresses in arb. frame that derivs. of A vanish in the original frame.

On specialising to identity transf^n, $\Gamma^n_{ki} = 0$.
also $= 0$ for linear transf^n $\because$ Γ involves ∂^2/∂^2

$\therefore$ We have here a bad set-up. 1 frame has some Γ which vanishes & in which special property holds.

Now. Crucial Point. Introduce new structure into space.
Namely a Γ^n_{ki} field which transforms so that

$$A_{k;i} = A_{k,i} - \Gamma^n_{ki} A_n \quad \text{is a tensor.}$$

Now the Γ^n_{ki} are the affine connexion and the $A_{k;i}$ are the covariant derivative w.r.t. this connexion.

Tempting to take Γ^n_{ki} symmetrical in k, i. $\because$ prototype qty. involved something sym. in k, i.
From pt. of view of structure above, this is not necessary. Keep it general for a while (not the common practice in rel^y books, which take it sym. straight away).

Transformation law for Γ is $\quad \Gamma'^n_{ik} = \dfrac{\partial x'^n}{\partial x^l}\dfrac{\partial x^r}{\partial x'^i}\dfrac{\partial x^s}{\partial x'^k}\Gamma^l_{rs} + \dfrac{\partial x'^n}{\partial x^l}\dfrac{\partial^2 x^l}{\partial x'^i \partial x'^k}$

2nd term is indep. of Γ & prevents Γ being a tensor.
It can vanish in one frame without vanishing in all.

Properties of Γ

i) despite its non-tensority, symmetry of Γ^n_{ik} in i,k is invariant, $\because$ 2nd term in transfn. law is symmetrical.

ii) for same reason, skew part of Γ, $\Gamma^n_{\check{ik}}$ is a tensor. "Nuisance term" goes out.

iii) $\Gamma^n_{ik} - \hat{\Gamma}^n_{ik}$ is a tensor (2 affine connexions defined on one space)
In particular $\delta\Gamma^n_{ik}$ is a tensor.

iv) $\lambda\Gamma + \mu\hat{\Gamma}$ is also a connexion if $\lambda + \mu = 1$

Covariant derivative of contravariant vector.

We define: $\quad (A_k B^k)_{,i} = (A_k B^k)_{;i} \quad$ with product rule for cov. deriv.

To satisfy this cond^n $\quad B^k{}_{;i}$ must $= B^k{}_{,i} + \Gamma^k_{ni} B^n$

Derivatives of a tensor will follow $\because$ tensor behaves like product of vectors.

Derivatives of tensors. $T^{kl\cdots}{}_{pq\cdots;i} = T^{kl\cdots}{}_{pq\cdots,i}$

$+ T^{rl\cdots}{}_{pq\cdots}\Gamma^{k}_{ri} + \cdots$ sim. terms for each u'stts index of T

$+ - T^{kl\cdots}{}_{rq\cdots}\Gamma^{r}_{pi} - \cdots\cdots$ sim terms for each d'stts index of T

Index of diff: is always 2nd d'stts index of Γ

Other two on Γ are index of T we are operating on, u'stts or d'stts accordingly, and dummy index.

$\pm$ for u'stts or d'stts index being operated on.

En passant, $\delta^{k}{}_{l;m} = 0$.

If something's cov. derivative is $=0$, can say that it is "covariantly constant."

Diff: of density of weight 1. (will only consider wt. 1 henceforth)

We choose to define i) Product rule satisfied

ii) ε^{iklm} has derivative $=0$.

Then can show that $\mathfrak{T}^{\cdots}{}_{\cdots;i}$ has extra term $-\Gamma^{r}_{ri}\mathfrak{T}^{\cdots}{}_{\cdots}$

Can make symmetric part of Γ vanish at any arb. pt. by suitable choice of co-ord system.

Frequently useful in proving theorems. (Cov. deriv then $\to$ ord^y deriv. if Γ sym overall)

Co-ordinate system is then called "geodesic co-ordinates."

We seek co-ord syst. s.t. $\Gamma^{i}_{\underline{kl}} = 0$ at some pt, e.g. $x^k = 0$. ($\Gamma^{i}_{\underset{\vee}{kl}}$ is tensor remember)

Now $\Gamma'^{i}_{kl} = \dfrac{\partial x'^i}{\partial x^r}\dfrac{\partial x^s}{\partial x'^k}\dfrac{\partial x^t}{\partial x'^l}\Gamma^{r}_{st} + \dfrac{\partial x'^i}{\partial x^m}\dfrac{\partial^2 x^m}{\partial x'^k \partial x'^l}$

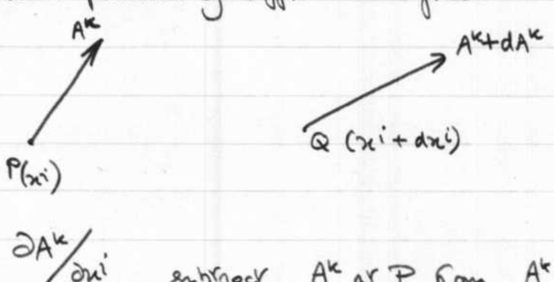
Suppose co-ord. transf: is s.t. $x^k = x'^k + \frac{1}{2}a^{k}_{lm}x'^l x'^m$ $(+\cdots\cdots)$ arbitrary higher powers.

At origin, let $x^k = x'^k = 0$. $\Gamma'^{i}_{kl} = \Gamma^{i}_{kl} + a^{i}_{kl}$ Whatever power of transf: we choose.

Therefore $\Gamma'^{i}_{\underline{kl}} = 0$ if $a^{i}_{kl} = -\left(\Gamma^{i}_{\underline{kl}}\right)_{x^k=0}$ (a^{i}_{kl} must be sym.)

Parallel Transfer.

Reconsider problem of diff.^{n} vector field.



To form $\partial A^k/\partial x^i$, subtract A^k at P from $A^k + dA^k$ at $Q \to dA^k$, not a vector. $= \dfrac{\partial A^k}{\partial x^i}dx^i$

$\therefore$ not a tensor. vector

What we really want to do is to subtract two vectors at same pt. Have some vector at Q which in some sense is the same vector that we had at P. "Parallel" & "of same length".

We must stipulate what change in cpts. of A^k when we go from $P \to Q$ counts as "no change" for the present purpose. Can't be literally "no change", $\because$ this is not invariant. No change in 1 co-ord mpt $\not\Rightarrow$ "no change" in another.

Let this "unchanged" vector be $A^k + \delta A^k$. δA^k is not a vector.

But now we have 2 vectors at same Q, $\underset{\uparrow}{(A^k + dA^k)} - \underset{\uparrow}{(A^k + \delta A^k)}$ is a vector.

Change due to vector field — Change due to pos. in manifold.

δA^k will depend on A^k and dx^i. Assume linearity.

$$\delta A^k = -\gamma^k_{li} A^l dx^i$$

The γ's are then an array of 64 coeffts. of proportionality. Fns. of position.

γ's obv. not tensors, $\because$ A^l, dx^i ~~tensors~~ vectors, δA^k not a vector.

Deduce transformation law by requiring covariance of our definition of δA^k.

Find that $\gamma^k_{li} = \Gamma^k_{li}$

$A^k + \delta A^k = A^k - \Gamma^k_{li} A^l dx^i$ is called the parallely transferred vector.

Note that this is vector at Q, A^k def. at P, so that non-tensor character of Γ^k_{li} is o.k. in a vector!

This was first done by embedding n-space in $(n+1)$ Euclidean space. Realised later that it's an intrinsic process.

Can then construct cov. derivative as

$$A^k_{;i} = \lim_{dx^i \to 0}\left[\frac{A^k + dA^k - A^k - \delta A^k}{dx^i}\right]$$

$$= \lim\left[\frac{dA^k - \delta A^k}{dx^i}\right]$$

$$= A_{k,i} + \Gamma^k_{li} A^l \qquad \text{as before.}$$

Similarly for tensors.

The Γ's establish linear 1:1 mapping of vectors at neighbouring pts.

Pts. are now "connected" $\because$ vectors can be mapped between them. Affine means "linear 1:1" really.

Note that no particular reason in this approach for symmetric Γ.

<u>Integrability of Parallel Transfer.</u>

Can repeat differential displacement $P \to Q$ distant.

Can we go $P \to$ same Q by diff. routes & get same vector? Alternatively, can we take A^k round closed loop & get back to same vector at P

If the answers are "yes", call the affine connexion "integrable".

/ N & S condns. for compatibility is that parameter s defined on any path by $ds^2 = g_{ik}dx^i dx^k$ is affine.

For such s i) $g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = 1$

ii) $\frac{dx^l}{ds}\left(\frac{dx^i}{ds}\right)_{;l} = 0$ (Affinity)

Hence $g_{ik;l}\frac{dx^i}{ds}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$ at any pt. for all paths thru that pt.

This $\rightarrow$ $g_{ik;l} + g_{kl;i} + g_{li;k} = 0$.

If $g_{ik;l} = T_{lik}$, where $T_{lik} = T_{lki}$ & $0 = T_{lik} + T_{ikl} + T_{kli}$, metrics are compatible.

$$\therefore\ g_{ik,l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = T_{lik} \quad \times -\tfrac{1}{2}$$

$$g_{kl,i} - g_{ml}\Gamma^m_{ki} - g_{km}\Gamma^m_{li} = T_{ikl} \quad \times \tfrac{1}{2}$$

$$g_{li,k} - g_{mi}\Gamma^m_{lk} - g_{lm}\Gamma^m_{ik} = T_{kli} \quad \times \tfrac{1}{2}$$

Add $\rightarrow$ $g_{ml}\Gamma^m_{ik} = \frac{1}{2}(g_{kl,i} + g_{li,k} - g_{ik,l}) - \frac{1}{2}(T_{ikl} + T_{kli} - T_{lik})$

Define inverse of g_{ik} to be g^{ik} s.t. $g^{ik}g_{lk} = \delta^i_l$ Then g^{ik} is a tensor (normalised cofactors)

$$\Gamma^s_{ik} = \left\{ {s \atop ik} \right\} + g^{sl}T_{lik} \quad \text{is most general symmetric } \Gamma$$

where the Christoffel 3-index symbols are defined by $\left\{ {s \atop ik} \right\} = \frac{1}{2}g^{sl}(g_{li,k} + g_{lk,i} + - g_{ik,l})$

40 Γ's then depend on 10 g_{ik} & 20 T_{lik}. 40 → 30. Still need to reduce no. of indep vars.

We want i) Only 10 variables as $\exists$ 10 T_{kl} (energy-mtm. tensor of matter has only 10 cpts)

ii) R_{kl} symmetric, so that we can have $R_{kl} \propto T_{kl}$.

Most easily reduced to 10 vars. if take $T_{lik} = 0$. This turns out to satisfy (ii) also.
Then $\Gamma^s_{ik} = \left\{ {s \atop ik} \right\}$, $g_{ik;l} = 0$.

This is the case of Riemannian Geometry. Assumed by Einstein ab initio. See that it is not the most general geometry on a curved manifold. Could excer [?] otherwise.
$g_{ik;l} \to 0$ means that length of any vector is preserved in parallel transfer.

Elementary Props. of R.G.
i) $g^{ik}{}_{;l} = 0$. Pf: $g^{ik}g_{lk} = \delta^i_l$ by def.

$$g^{ik}{}_{;m}g_{lk} + g^{ik}g_{lk;m} = \delta^i_{l;m} = 0$$

$$\therefore\ g^{ik}{}_{;m}g_{lk} = 0. \qquad \therefore\ g^{ik}{}_{;m}g_{lm}g^{ls} = 0$$

$$g^{ls}{}_{;m} = 0$$

ii) $(\sqrt{-g})_{;l} = 0$ ($\sqrt{-g}$ is a scalar density)

Pf: $\partial g/\partial x^l = g g^{ik}\frac{\partial g_{ik}}{\partial x^l} = g g^{ik}\{\Gamma^m_{il}g_{mk} + \Gamma^m_{kl}g_{im}\}$, as $g_{ik;l} = 0$.

$$= 2g\Gamma^m_{ml}$$

Now $(\sqrt{-g})_{;l} = \partial/\partial x^l(\sqrt{-g}) - \sqrt{-g}\,\Gamma^m_{\ ml}$ Signs?

$= \partial/\partial x^l(\sqrt{-g}) + \frac{1}{2\sqrt{-g}}\,\partial g/\partial x^l$

$= 0$

iii) $R_{kl} = 0.$ $\because R_{kl} = \frac{1}{2}\left(\Gamma^m_{\ lm,k} - \Gamma^m_{\ km,l}\right)$

$= \frac{1}{4g}\left(\frac{\partial^2 g}{\partial x^k \partial x^l} - \frac{\partial^2 g}{\partial x^l \partial x^k}\right) = 0.$

Raising & Lowering Indices of Tensors.

Using g^{ik} & g_{ik} can associate with any $T^{ab\cdots}_{\ \ \ \ pq\cdots}$ other tensors.

$$T_a^{\ b\cdots}{}_{pq\cdots} = g_{al}T^{lb\cdots}{}_{pq\cdots\cdots}$$

$$T^{ab\cdots p}{}_{q\cdots\cdots} = g^{pl}\,T^{ab\cdots}{}_{lq\cdots\cdots}$$

These associated tensors are considered conventionally to be diff. forms of same tensor. Denoted by same "Kernel" letter.

Usual to space indices. i.e. $T^a{}_b$ not T^a_b. Where does a go when lowered otherwise?

Props. of raising & lowering.

i) Raise × Lower = Same. $\because$ 2 g's × to → δ.

ii) Dummy indices can be raised & lowered simultaneously.

$$T^{ab\cdots}{}_{aqr\cdots\cdots} = g^{al}\,T_l^{\ b\cdots m}{}_{qr\cdots}\,g_{ma}$$

$$= T_a^{\ b\cdots a}{}_{qr\cdots}$$

We can also "densify" a tensor by ×ing with $\sqrt{-g}$, a scalar density.

Since g^{ik}, g_{ik}, $\sqrt{-g}$ all have cov. deriv. $=0$, they parallely transfer into themselves. Hence raising, lowering & "densifying" are invariant under $\|^l$ transf.

Th: Raising & Lowering commutes with cov. diffⁿ: e.g. $g_{ik}B^{lk}{}_{;m} = B_i^{\ k}{}_{;m}$

$$LHS = (g_{il}B^{lk})_{;m} - \underbrace{g_{il;m}}_{=0}B^{lk}$$

$$= B_i^{\ k}{}_{;m}$$

Note that if we raise indices on $g_{ik} \to g^{ik}$ $\because g_{ik}g^{li}g^{kp} = g^{lp}$

$$\delta g^{is}g_{sk} + g^{is}\delta g_{sk} = 0$$

$\times g^{kl} \to$ $\delta g^{il} = -g^{is}g^{kl}\delta g_{sk}$ $\therefore$ Assoc. tensor of δg takes − sign.

Length of vector $\quad \left.\begin{aligned} A^2 &= g_{ik}A^iA^k \\ &= g^{ik}A_iA_k \\ &= A^iA_i \end{aligned}\right\} \equiv r.$

If g_{ik} is indefinite, $A^iA_i = 0$ possibly even if $A_i \neq 0$.

$g_{ik}A^iB^k \to \angle$ bet. 2 vectors if A, B normalised. $g_{ik}A^iB^k \to \cos\theta$.

$= 0 \to$ orthogonality.

Curvature tensor $\qquad R^i_{klm} \qquad g_{is} R^i_{klm} = R_{sklm}$

$$R^i_{klm} = \tfrac{1}{2} g^{is}\left(-g_{ls,km} + g_{kl,sm} + g_{ms,kl} - g_{km,ls}\right)$$

Then $R_{sklm} = \tfrac{1}{2}\left(\qquad\qquad\qquad\qquad \right)$

Further symmetry props.

i) R_{sklm} is skew in sk as well as lm

ii) symmetric in the pairs $sk \longleftrightarrow lm$.

Sym. props $\rightarrow$ 20 indep. cpts.

In general $= \binom{n}{2}\left\{\binom{n}{2}+1\right\} - \binom{n}{4} = \frac{1}{12}n^2(n^2-1)$ for n-space.

No. of vars. i) No. of dimensions $\rightarrow$ 2 3 4 5
ii) No. of indep. cpts of R_{abcd}. $\rightarrow$ 1 6 20 50

$n=2$. Curv. tens. cptely det by its contractions $= R = g^{kl}R_{kl}$

Unit tensor for tensors with symmetry of R_{abcd} is $(\delta^n_l g_{ki} - \delta^n_k g_{il})$

$$R^n_{ikl} = \tfrac{1}{2}\left(\qquad\qquad \right) R$$

$n=3$. R_{kl} (1st contraction) determines R cptely.

$$R^n{}_{ikl} = \delta^n{}_l R_{ki} - \delta^n{}_k R_{il} + g_{ki}R_l{}^n - g_{il}R_k{}^n - \tfrac{1}{2}(\delta^n{}_l g_{ki} - \delta^n{}_k g_{il})R$$

4 dimensions is least no. of dim. in which Riemann tensor not cptely det. by its contractions. Least no. in which $R_{kl}=0 \not\Rightarrow R^n{}_{ikl}=0$.

Bianchi Identities $\qquad R_{rsmn;t} + R_{rsnt;m} + R_{rstm;n} = 0$

Contract $\times\, g^{rn} g^{sm} \rightarrow \quad g^{sm}R_{sm;t} - g^{sm}R_{st;m} - g^{rn}R_{rt;n} = 0.$

$$R_{;t} - 2R^n{}_{t;n} = 0$$

$$(R^n{}_t - \tfrac{1}{2}R\delta^n{}_t)_{;n} = 0$$

Define $G^n{}_r = R^n{}_r - \tfrac{1}{2}R\delta^n{}_r \qquad$ Einstein Tensor.

Conserved $\because$ its cov. dgce vanishes. Will become Energy – Mtm. tensor. $G^n{}_{t;n} = 0$.

Paths as Geodesics.

Rel: bet metric + path?

Geodesic is defined as curve whose length is stry for arb. small varns. of curve while end pts are fixed.

$$\delta \int_A^B ds = 0. \quad = \delta \int_A^B \left(\sqrt{g_{ik} \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda}} \right) d\lambda$$

$$\frac{d^2x^r}{ds^2} + \Gamma^r_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = 0 \qquad \text{Geodesic eqn.}$$

Normally use ds for $d\lambda$. But not for null path $ds = 0$. Must use another parameter.

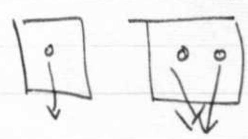
Therefore $\rightarrow$ $g_{mn} \frac{dx^m}{du} \frac{dx^n}{du} = \text{const.} = 0$ for null geodesic. $= \pm 1$ for space-like / time-like if normalised.

Tangent vector to null geodesic is self-orthogonal. Not self-orthog. for gen. geodesic.

Curve that is null everywhere $\not\Rightarrow$ null geodesic. Variation strategy. is paramount.

Geodesic Deviation

Can't tell if space is curved unless return to starting-pt. Cannot tell curvature for 1 geodesic (can't tell grav' field i.e. by dropping 1 particle.)



Sep^n of 2 geodesics $\rightarrow$ curvature.

Sphere. Flat. ← Different!

Use u as arc length meas. along geodesic.

Draw Let $x^r = x^r(u,v)$

AA' orthog' to geodesics.

A, A' have same u

P, P' " " "

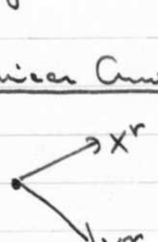
Let $PP' = \eta^r = \frac{\partial x^r}{\partial v} dv$

Absolute Deriv. $\frac{\delta A}{\delta u} = A^i_{\ ;k} \frac{dx^k}{du}$ Note: s for u → unit tangent. Preferable. Except u if null again.

Find. $$\frac{\delta^2 \eta}{\delta s^2} + R^r_{\ smn} p^s \eta^m p^n = 0 \qquad p^r = \frac{dx^r}{ds} \quad \text{(Synge + Schild)}$$

Only if $R = 0$ does η become linear.

Riemannian Curvature.



Let Z^r be $aX^r + bY^r$ a & b scalars.

Define the pencil of vectors lin. dep^t on X & Y. Define elementary 2-space at the pt.

Def^n: Riem. Curv. of the el. 2-space is $K = R_{rsmn} X^r Y^s X^m Y^n$
where X^r, Y^r are orthog unit vectors. (Space-like vectors).

Can be shown that
K is indepr. of choice of base vectors. (S & S'). provided that vectors do define same surface!
If manifold in 2-space, K is unique. $\to R \to$ curvature cpletly def. by scalar.

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$$

Cf. Gaussian curvature.
Any 2-surface embedded in E_3 has 2 princ. radii of curvature R_1 & R_2 say.

$$G \text{ def} = \frac{1}{R_1 R_2}$$

G proved to be intrinsic props. of surface $\quad G = \lim_{S \to 0} \left(E/S \right)$ $\quad$ S = area of geodesic 4-lateral; E = xs of sum of ∠s over 4 ∟s

$G > 0$ +ve curvature (ellipsoidal) $\qquad G < 0$ -ve curvature (hyperbolic)

If take a vector by ∥ transfer around a small geodesic 4-lateral, transported vector differs from original by ∠ KS.
But the vector makes const. ∠ with geod. under ∥ transfer, so that $KS = E$. Proved in S & S'.

Meaning of Riemannian Curv. in n-dim.
Consider 2 geodesics. emanating from a pt. O

$$\frac{\delta^2 \eta^i}{\delta s^2} + R^i{}_{jkl}\, p^j p^l \eta^k = 0. \qquad (A)$$

Where $\frac{\delta \eta^i}{\delta s} = \eta^i{}_{;k} \frac{dx^k}{ds} \qquad p^k = \frac{dx^k}{ds}$

Let u^i be unit vector along η^i, & $\eta = |\eta^i|$. $\qquad \eta^i = \eta u^i$

Subst. in (A) $\to$
$$u^i \frac{d^2\eta}{ds^2} + 2\frac{d\eta}{ds}\frac{\delta u^i}{\delta s} + \eta \frac{\delta^2 u^i}{\delta s^2} + \eta R^i{}_{jkl}\, p^j p^l u^k = 0. \qquad (B)$$

$$\times u_i \quad \frac{d^2\eta}{ds^2} + 2\frac{d\eta}{ds} u_i \frac{\delta u^i}{\delta s} + \eta u_i \frac{\delta^2 u^i}{ds^2} + \eta K = 0. \qquad (C)$$

Now $u_i u^i = 1$. $\therefore \frac{\delta u^i}{\delta s} u_i = 0. \quad$ & $\quad u_i \frac{\delta^2 u^i}{\delta s^2} = -\frac{\delta u^i}{\delta s}\frac{\delta u_i}{\delta s} \qquad (D)$

Expand η in p-series about O

$$\eta(s) = \underset{\underset{0}{\parallel}}{\eta(0)} + \underset{\underset{\vartheta}{\parallel}}{\frac{d\eta}{ds}(0)}\, s + \frac{1}{2}\frac{d^2\eta}{ds^2}(0)\, s^2 + \frac{1}{6}\frac{d^3\eta}{ds^3}(0)\, s^3$$

Let $s \to 0$ in (C) $\quad \frac{d^2\eta}{ds^2} = 0. \qquad$ Let $\eta \to 0$ in (B), from term 2, $\frac{\delta u^i}{\delta s}$ must $\to 0$.

$\therefore \frac{\delta^2 u_i}{ds^2} \to 0$ fr (D)

Then (C) becomes $\frac{d^2\eta}{ds^2} \cdot \frac{1}{\eta} + K = 0.$

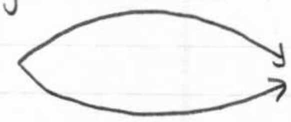
$$\therefore \frac{d^3\eta}{ds^3} = -K\frac{d\eta}{ds} - \eta \underbrace{\frac{dK}{ds}}_{0 \text{ with } \eta}$$

$$\frac{d^3\eta}{ds^3} \therefore \to -\kappa\theta \text{ as } \eta = 0.$$

$$\eta(s) \therefore = \theta(s - \tfrac{1}{6}\kappa s^3) \text{ in er.}$$

If $\kappa = 0$. Straight line.

κ = +ve



κ = −ve

End of Diffe. Geometry per se.

Gravitational Field Eqn.

$R_{kl} = 0$ field without matter

$R_{kl} = \kappa T_{kl}$ matter present. Suggested earlier, but 3 snags.

Snag ① — R_{kl} not sym. Now $R_{kl} = R_{lk}$ arranged

② — R_{kl} only defined & $T^{kl} = \rho v^k v^l$ Now with metric we can raise & lower indices.
$\therefore$ ok, can make T_{kl}.

③ — 40 Γ's before. Now have Γ's det by 10 g_{ij} thru metric.

$T^{ij}{}_{,j} = 0$ in S.R.

G.R. expect cov. cons. law, such as $T^{ij}{}_{;j} = 0$.

But G_{kl} (Einst. tensor) $= R_{kl} - \frac{1}{2} R g_{kl} = \kappa T_{kl}$ $G^{kl}{}_{;l} = 0$. (Bianch. id.)

In empty space $R_{kl} - \frac{1}{2} R g_{kl} = 0$.

$\times g^{kl} \to R - 2R = -R = 0$.

$\therefore$ In empty space $R_{kl} = 0$.

Can include cosmological term $+ \lambda g_{kl}$. Leave to Sciama. Doesn't affect cons.

These are only FE's sat. i) no higher than 2nd deriv. of g_{ik}
ii) lin & homog. in 2nd deriv.
iii) Div. vanishes identically.

But i) Not linear in 1st deriv.
ii) Can make arb. co-ord transfm. dep on 4 functions so that solutions must contain 4 arb fns.
i.e. they are not 10 indep eqns. Must exist 4 relns.
These are the Bianchi identities.

$\therefore$ Expect that Bianchi identities could be obt. from co-ord inv. of F.E's.

Or is the "Bnt's" which give rise to trouble if try to quantise the ∇?

<u>Comparison with Newton.</u>

GR described by $R_{ij} - \frac{1}{2}Rg_{ij} = KT_{ij}$

$$\frac{d^2x^r}{ds^2} + \Gamma^r_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0$$

1st approx. Flat. $g_{ij} = \eta_{ij} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}$

All Γ's $=0$.

2nd approx. $g_{ij} = \eta_{ij} + h_{ij}$, $h_{ij}^2 = 0$.

Assume small velyp. $\frac{dx^A}{ds} = 0 \quad A = 1,2,3$

$$\frac{dx^4}{ds} \sim 1$$

Also $\frac{\partial h_{ij}}{\partial x^4} \doteq 0$ Grav! field not o. t-const.

Then Eq.n of motion is $\frac{d^2x^r}{ds^2} = -\frac{\partial}{\partial x^r}\left[\frac{1}{2}h_{44}\right]$

Then h_{44} corresponds to the Newtonian potential ϕ. Does $\frac{1}{2}h_{44}$ satisfy $\nabla^2\phi = \rho$?

For this approx.

$$R_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma} + \frac{\partial^2 g_{\rho\sigma}}{\partial x^\nu dx^\nu} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^\rho \partial x^\nu} - \frac{\partial^2 g_{\nu\rho}}{\partial x^\mu \partial x^\sigma}\right)$$

$$\to \frac{1}{2}\eta^{\sigma\rho}\frac{\partial^2 h_{\mu\nu}}{\partial x^\sigma \partial x^\rho} + \frac{1}{2}\frac{\partial}{\partial x^\mu}\left[\frac{\partial h}{\partial x^\nu} - 2\frac{\partial h^\rho_\nu}{\partial x^\rho}\right]$$

for $h = \eta^{\mu\nu}h_{\mu\nu}$

Choose coord syst s.t. $h_{\mu\nu}$ small & $\frac{\partial h}{\partial x^\nu} - 2\frac{\partial h^\rho_\nu}{\partial x^\rho} = \frac{\partial}{\partial x^\alpha}\left[h_{\alpha\mu} - \frac{1}{2}h\eta_{\alpha\mu}\right] = 0$.

This <u>is</u> possible.

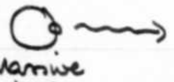
Then have $\Box^2 h_{\mu\nu} = 2R_{\mu\nu}$

$$\text{or } \Box^2\left(h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\right) = 2\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = 2KT_{\mu\nu}$$

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$$

$$\left.\begin{aligned} \Box\gamma_{\mu\nu} &= 2KT_{\mu\nu} \\ \frac{\partial \gamma_{\mu\nu}}{\partial x^\nu} &= 0. \end{aligned}\right] \text{ Maxwell's Eqns \& Lorentz cond.}$$

$\to$ Soln $\gamma = \frac{1}{2\pi}\int\left[\frac{KT_{\mu\nu}}{r}\right]dV$ consistent with $\gamma_{\mu\nu,\nu} = 0$.



Massive body

$\delta\nu = -\frac{m}{r}\nu_0.$

3 tests of G.R. (i.e. beyond Newtonian th?)

i) Einstein Red Shift. For sun, $\delta\nu \sim -2.10^{-6}\nu_0$.
At $\lambda 4000$, $\delta\lambda \sim .0082$Å. No difficulty in measuring this, but confusion of other effects. Doppler shifts, etc. Red shifts have been observed, & amount of red shift is fn. of solar latitude & longitude. $\delta\lambda \sim$ Einstein near limb $\sim 2\delta\lambda_E$ near centre. No clear-cut verification.

White dwarfs. Large m/r $\because$ of small r. Sirius B also problematic.
i) radius uncertain, ii) must get Sirius B against Sirius A. Again nothing unambiguous.

Mössbauer Effect. Fe radioactive $\to$ γ ray whose ν is measurable to 10^{-15}.
Pound & Rebka. $\Delta\phi/\phi \sim 10^{-15}$. Advantage of more complete control. Verified $\pm$ 10%.

Energy method. ΔE must come from $\Delta\nu$. & $m = E/c^2$. $\to$ Einstein formula.
Have still to assume that energy of a photon does gravitate. $\therefore$ Does involve Principle of Equivalence, altho' agt. Newtonian - Quant Mech.

Crucial test 2. Geodesics.
On view that null-geodesic in S.R. describes light path, take null-geodesic in G.R. too.

$$\frac{d^2x^r}{du^2} + \Gamma^r_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0$$

$$\therefore \int g_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \quad \text{i.e. } ds^2 = 0. \qquad \to \text{3-surface (light cone, null-cone).}$$

Take simple case $\quad ds^2 = f^2 dt^2 - d\sigma^2 \qquad f(\text{posn. only}) \quad d\sigma^2 = \text{3-dim. spatial metric.}$
$= \gamma_{ik}\,dx^i dx^k$

Static metric.

i.e. $g_{44} = f^2$, $g_{4i} = 0$, $g_{ik} = \gamma_{ik}$ $\quad i,k = 1,2,3.$ $\quad f, \gamma_{ik}$ indep. of x_4.

Then $ds^2 = 0 \implies \frac{d\sigma}{dt} = f.$ $\quad$ i.e. f is "vel." of light in this co ord. syst.
If $f \neq 1$, we have a "refractive index" effect.
$f = 1 - m/r$ for ex., from weak field approx^n, $\mu \sim 1/1 + m/r$

$\therefore$ In presence of a lump, space has variable refractive index $\to$ deviation.
$\to$ Hyperbolic path near the sun.
Wait for total eclipse to make nearby stars observable & cpre with 6 months later.

$$\text{Total asymptotic deflexion } \alpha = \int_{-\infty}^{+\infty}\frac{1}{f}\frac{\partial f}{\partial x}\,dy \quad (\text{by Huyghens}) \quad \to \quad 4m/R.$$

Quasi-Newtonian $\to 2m/R$.
For grazing incidence. $R = R_\odot$, $\alpha = 1.75''$, mod if XS >> XE. Not yet practible for planet.
$\because$ of local δ subtracting from α.

Eddington I. Nearer 4 than 2
Later ones. also nearer 4 than 2. But statistical errors are still highly detectable. A deflexion has been measured, but may have been influenced by foreknowledge of answer to converge the statistics.

Perihelion of Mercury.

An exact sol.ⁿ of field eqns. Note that these are rare ∵ we can't even formulate the problem properly until we know the geometry, which is part of the solution!

Generally high symmetry restricts the metric, then can adjust metric to give the problem!

i) Swarzschild solⁿ.

Point mass m.

Einstein used successive approxⁿˢ. Weak field sol.ⁿ not good enough ∵ data so good (10^3 years of careful astronomy!). Swarzschild (1916) achieved exact soln tho.

Assume spherical symmetry.

Then ∃ $\quad ds^2 = e^{\lambda} dr^2 + e^{\mu} d\Omega^2 - e^{\nu} dt^2 + 2a\,dr\,dt \qquad \lambda = \lambda(r,t)$, $\mu = \cdot\cdot$, $\nu\ \cdot\cdot\cdot$, $a\ \cdot\cdot\cdot\cdot\cdot$

Assume system static, $a = 0$. since t-reversal makes no differ.

Put $r' = e^{\mu} r^2$.

Then, dropping primes ∵ we don't need them $\Rightarrow ds^2 = e^{\lambda} dr^2 + r^2 d\Omega^2 - e^{\nu} dt^2$

Also, if static, λ is $\lambda(r)$ only, ν is $\nu(r)$ only.

Now insert the metric, with unknown λ & ν into field eqns.

Calculate $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ for this metric and equate $K T_{\mu\nu}$ (take units with $K = 1$)

$$T^1_1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$T^2_2 = T^3_3 = -e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'-\lambda'}{2r}\right)$$

$$T^4_4 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$T^{\alpha}_{\beta} = 0 \quad \alpha \neq \beta. \qquad \text{Cosmological const.} = 0.$$

Have not yet specified pt. mass. This would apply to any sph. sym. distribⁿ of en. & mtm.
Outside the particle we require $T^{\alpha}_{\beta} = 0$ now. → set of differ. eqns.

1st & 3rd $\rightarrow \lambda' = -\nu'$. 2nd $\rightarrow \nu'' + \nu'^2 + \frac{2\nu'}{r} = 0.$

$$\Rightarrow e^{\nu} = A + B/r$$

Chose A & B to satisfy Bdry Condns. 1. ∞ S.R. should hold, ∴ $A = 1$
(no galaxies, no nothing — no cosmology)

Now B must have something to do with the mass, ∴ put $B = -2m$!

$$e^{\nu} = 1 - \frac{2m}{r}$$

Similarly, $\quad e^{-\lambda} = 1 - \frac{2m}{r}$

$$\therefore\ ds^2 = \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\Omega^2 - \left(1 - \frac{2m}{r}\right) dt^2$$

SWARZSCHILD SOLUTION

Could have used this metric to → Einstein red shift & light-bending if we'd wanted to be fancy.
To cf. previous approxⁿ, change to isotropic co-ords.

$$r = \left(1 + \frac{m}{2r}\right)^2 r'$$

$$ds^2 = \left(1 + \frac{m}{2r'}\right)^4 (dr'^2 + r'd\Omega^2) - \frac{(1 - m/2r')^2}{(1 + m/2r')^2} dt^2$$

Now put x, y, z thru usual flat transf.ⁿ

$$ds^2 = \left(1 + \frac{m}{2r}\right)^4 (dx^2 + dy^2 + dz^2) - \frac{(1-m/2r)^2}{(1+m/2r)^2} dt^2$$

Chosen this form ∵ we get all of space part ×ed by the same factor (hence isotropic).

In these coords. coord vel. of light & co-ord length of rigid rod indep.ᵗ of orientation.
For small m/r

$$ds^2 = (1 + 2m/r)(dx^2 + dy^2 + dz^2) - (1 - 2m/r)dt^2 \qquad \text{W.F.A.}$$

To deal with mercury, need higher approx:
Regard planets as test particles. Neglect their fields (perturbⁿ, etc.)
Basic physical hypothesis – they will run along geodesics, being test particles.
Calculate Christoffel symbols for exact Sw. solⁿ & insert into eqns. of motion.

Results i) space-orbit is plane
ii) Newtonian eqns would be $\frac{d^2u}{d\phi^2} + u = m/h^2$, $r^2 \frac{d\phi}{dt} = h$, $u = 1/r$

$$G.R. \longrightarrow \quad \frac{d^2u}{d\phi^2} + u = m/h^2 + 3mu^2, \quad r^2\frac{d\phi}{ds} = h$$

h = const. in both O's (area rule).
The perturbing term is $3mu^2$. Compare it with m/h^2, ratio $3u^2h^2 = 3\left(r\frac{d\phi}{ds}\right)^2$
$dt \sim ds$, ∴ in perturbⁿ take $dt = ds$ ∵ perturbⁿ itself small. $\rightarrow 3v^2/c^2$. restoring c.
Earth $3v^2/c^2 \sim 3.10^{-8}$. Now we know devⁿ from Newt. small, ∴ Newtonian motion to a high ° of approx: ∴ m = mass!

Have succeeded in getting co-ordinate system identifiable with Galilean. This is possible ∵ curvature is small. Space nearly flat. Even at solar surf, $2m/r$ is only 4.10^{-6}
Not gen. possible. Can get a prediction independent of co-ord system?
This shows how devⁿ of gμν from Galilean vals. may be negligible metrically but vital gravitationally.

Effect of perturbing term on planetary orbit.
Perihelion rotates – in one revolⁿ rotn is $\Delta\omega = \frac{6\pi m^2}{h^2}$

2° of freedom of motion of particle in central field of force. Think of it as 2 oscillations in 2 perpendicular directions. General law of force → 2 frequencies. Can get degeneracy due to some symmetry or simple law of force. Without degeneracy ↛ closed orbit.
-2 law → degeneracy from 3d → plane & then further degeneracy → closed orbit.
When perturbing term introduced the degeneracy is removed (as in Q. Mech.) & ↛ closed orbit.
∵ it's a <u>small</u> perbⁿ can describe in terms of ellipse that rotates.
See that rotⁿ is <u>not</u> a relativistic effect. Any non-I.S. law will → rotⁿ of perihelion in some manner. It is quantitative description that must be correct. Perturbations of other planets → rotⁿ of p/helion. After correction for these, there was <u>still</u> a residual rotⁿ
All other attempts to explain this came to some grief.

Einstein GT not tailor-made for perihelia! Remarkable that it gets quantitative results right when only very general philosophical ideas at outset. Other GT's anyway rigged for perihelion-catching!

		$\delta\omega$ GT / cent	$e\delta\omega$	$\delta\omega$ Obs
Observations	Mercury	43".03 ± .03	8.847	42".5 ± 0.94
	Venus	8".63	0.059	
	Earth	3".84	0.064	4".6 ± 2".7
	Mars.	1".35	0.126	

Mars GT would perhaps help. Space probes in high-ε orbits near the Sun?

If cosmological λ introduced, Sw. soln. becomes: $1-\frac{2m}{r} \to 1-\frac{2m}{r}-\frac{\lambda}{3}r^2$.
λ-term → add. force over & above grav. force, attractive or repulsive as $\lambda \pm$.
As $r <$, λ-term becomes dominant.
Rough estimate of Σ on λ, $r_c < \sqrt{3/\lambda}$. from $\frac{\lambda}{3}r^2 \sim 2m/r$
r_c must be $> 10^{27}$ at least.

∴ λ must be $< 3.10^{-54}$ → negligible effects in Solar Syst.

Swarzschild Radius.

At $r = 2m$, metric goes $\updownarrow$!

For sun, $r_{Sw} \sim$ 1.5 km
electron 10^{-55} cm.

2 sorts of singularity in s-t. manifold.
i) $R \to \infty$. Happens at $r = 0$ here. This is ∵ of δ-fn. mass put in.
ii) arises from attempt to impose 1 co-ord syst. on whole business (like sphere)

Cannot fill all space with one co-ord syst. Attempt to do so has resulted in the bad behaviour of Sw. radius. All the relevant invariant qtys. are reasonable on r_{Sw}. Essentially a topological problem. Topology is v. complicated in Sw. soln. Can continue metric analytically across the radius

Light ray emitted inside Sw. rad never gets out.
As mod. by you, takes ∞ time to collapse.
Co-ord time for collapse ∞.
But if you were on it, proper time of collapse is finite.

Birkhoff Thm.

Sph. sym empty space fieldeq. → Sw. Soln. uniquely.
Note: do not need static. Can prove this.
∴ sph. sym. pulsations do not disturb distant metric. Grav. waves not radiated from symmetric distribs. Exactly like classical e/dynamics.

Equations of Motion.

Conservation Eqns. $T^{\nu}_{\mu;\nu} = 0$. Density $\mathfrak{T}^{\nu}_{\mu} = \sqrt{-g}\,T^{\nu}_{\mu}$

Then $\dfrac{\partial \mathfrak{T}^{\nu}_{\mu}}{\partial x^{\nu}} - \frac{1}{2}\mathfrak{T}^{\alpha\beta}\dfrac{\partial g_{\alpha\beta}}{\partial x^{\mu}} = 0$ ($\mathfrak{T}^{\alpha\beta}$ sym) upon manipulation.

In S.R. Cons. Eqn is dgce. of $T=0$.
∴ We have that energy-mtm is not conserved. i.e. energy can be transferred to body from grav! field. Let go of something → it falls & accelerates!

Know in S.R. that force density $f^{\mu} = \partial T^{\mu\nu}/\partial x^{\nu}$. ∴ Equate whole added term to force density.
Introduce Maxwell stress tensor $T'^{\mu\nu}$ of magnetic field s.t. its divgce is $-f^{\mu}$. in rel. e/dynamics
Suppose that

$$T^{\mu\nu} = T^{\mu\nu} + T'^{\mu\nu}$$

$$\text{Divgce of } T^{\mu\nu} = D(T^{\mu\nu} + T'^{\mu\nu}) = f^{\mu} - f^{\mu} = 0.$$

∴ The idea is familiar in S.R. //

Analogously. suppose grav! field has energy tensor whose dgce → force density from above.
Can't strictly be so ∵ we are in densities in grav! eqn. Hetlo! eqns are wholly covariant, they are not partially vectorial.

Approximate. System moving slowly. g_{44} most important term.
Then the "quasi-force" field has the form $-\frac{1}{2}\dfrac{\partial g_{44}}{\partial x}$, $-\frac{1}{2}\dfrac{\partial g_{44}}{\partial y}$, $-\frac{1}{2}\dfrac{\partial g_{44}}{\partial z}$.

Derivable from a potl. $\Omega = -\frac{1}{2}g_{44}$ + const. of integration.

Choose const. s.t. $g_{44} = 1$ when $\Omega = 0$.

$$(1 - 2\Omega) = g_{44}$$

Got this before from eqns. of motion. Now we have not assumed that particles move on geodesics.
Implies that can deduce from cons. laws what are eqns. of motion. Field eqns → eqns of motion.

2 Schools of Thought. i) Einstein, Infeld, Hoffman.
↓
Motion & Relativity

Do not describe particle by energy-tensor, but world-line of particle is singular line in metric. Integrability condⁿ → eqn. of motion between singularities.
Field eqns force singularities to lie on geodesics. Detailed discussion difficult ∵ dealing with singularities. ∵ of non-linearity cannot separate field due to a particular body from that of rest of sources.

ii) Fock ⟶ ~"gᵀ of S-T & Grav."

Lumps described by energy-mtm tensor. Does not have technical problems of singularities.

Why is the theory so much more complicated than Maxwellian e/dynamics?
Have 4 Conservation Identities

Non-linearity → gravity acts on gravity. V. necessary if motion to be deduced from field.

Maxwell is linear, & requires additionally Lorentz law of force.
Suppose we have sol.ⁿ of M.i for charge moving i space.
another charge .. .. Field stated everywhere.
∴ M.i linear, there is a sol.ⁿ of 2 charges present, e/m field present the sum of the 2 fields i the 2 problems. ∴ Charges don't notice one another. Cannot deduce interaction bet charges from field eqns.

∴ Non-linearity essential. Necessary, but not sufft. Born-Infeld th?, prevented pr. charges having ∞ self-energy. But despite non-lin? of B-I eqns can't get eqns. of motion ∵ not enough conservation identities. Can't get 4 eqns. of motion from 1 consⁿ identity. Need 4 identities. ∴ G.R. ideally set up for this!

Action Principle in the Theory.

i) Integration by parts.

$$\mathcal{A}^k{}_{;k} = \mathcal{A}^k{}_{,k}$$

Suppose $I = \int (A^{\cdots}_{\cdots})(B^{\cdots}_{\cdots})_{;k}\, d\tau$

If I scalar, $(A^{\cdots}_{\cdots})(B^{\cdots}_{\cdots}) \cdot \mathcal{A}^k$

Then $(A^{\cdots}_{\cdots} B^{\cdots}_{\cdots})_{;k} = (A^{\cdots}_{\cdots})(B^{\cdots}_{\cdots})_{;k} + (A^{\cdots}_{\cdots})_{;k}(B^{\cdots}_{\cdots})$ ∵ we made it so.

$$I = \int (A \quad B \quad)_{;k} - \int (A \quad)_{;k}(B \quad)$$

$$= \int \mathcal{A}^k{}_{;k} - (A \quad)_{;k}(B \quad) = \int \mathcal{A}^k{}_{,k} - (A \quad)_{;k}(B \quad)\, d\tau$$

Now ∵ 1st term is ordy. dgce. can → surface integral.
Usually have varⁿ vanishing on a surface. ∴ $\int \mathcal{A}^k{}_{,k}\, d\tau \to 0$.

∴ $\int (A \quad)(B \quad)_{;k}\, d\tau = -\int (A \quad)_{;k}(B \quad)\, d\tau$.

Rather as in ord. ∫atn by parts in a varⁿ principle.

Non-trivial ∵ $; \neq ,$ for A or B. Only for $\mathcal{A}$ or $\mathcal{B}$.

Empty space. Want scalar density $\mathcal{L}$. $L = \int \mathcal{L}\, d\tau$.

What simple scalar densities can we construct s.t. this prop.?
i) $\sqrt{-g}$. No derivatives ↛ diff. eqn. for g_{ij}.

ii) R $\quad \sqrt{-g}R \quad (+\lambda\sqrt{-g}$ if desired)

But R depo. on 2nd derivs. of g_{ik} → field eqns. would involve 4th derivs. of g_{ik}.
But then 2nd deriv. terms occur as a dgce. But then when we vary, fix bdy. condns. on surface → Green Thᵐ → they contribute nothing.

Try then $L = \int \mathfrak{g}^{ik} R_{ik} \, d\tau$ $\qquad\qquad$ $\mathfrak{g}^{ik} = \sqrt{-g}\, g^{ik}$.

$$\delta L = \int [\delta \mathfrak{g}^{ik} R_{ik} + \mathfrak{g}^{ik} \delta R_{ik}] d\tau$$

$$\delta R_{ik} = -(\delta \Gamma^{\alpha}_{ik})_{;\alpha} + (\delta \Gamma^{\alpha}_{i\alpha})_{;k}$$

By the ~~integration~~ by parts rule we derived last lecture, if varn. vanishes on bdry.
Also then Riemannian ~~geometry~~ geometry, $\mathfrak{g}^{ik}{}_{;\alpha} = 0$.

$$\int \mathfrak{g}^{ik} \delta R_{ik} \, d\tau = 0.$$

$$\delta L = \int R_{ik} \, \delta \mathfrak{g}^{ik} \, d\tau$$

$\delta L = 0 \Rightarrow R_{ik} = 0 \quad \because \delta \mathfrak{g}^{ik}$ arbitrary. $\qquad$ Einstein's Eqns for empty space.

Matter present.

Lagrangian is $\sqrt{-g} R + \mathcal{L}_{matter}$.

Vary w.r.t. g^{ik} $\qquad \delta \mathfrak{g}^{ik} = \sqrt{-g}\, \delta g^{ik} - \frac{1}{2}\sqrt{-g}\, g^{ik} g_{jl} \delta g^{jl}$

$$\frac{\delta R\sqrt{-g}}{\delta g^{ik}} = \sqrt{-g}\,(R_{ik} - \tfrac{1}{2} R g_{ik})$$

$\therefore$ Field ens. are $\qquad R_{ik} - \frac{1}{2} R g_{ik} = -\frac{\delta \mathcal{L}}{\delta g^{ik}} \cdot \frac{1}{\sqrt{-g}}$

Put $T_{ik} = \frac{\delta \mathcal{L}}{\delta g^{ik}}$ — symmetric tensor density, obviously.

$$R_{ik} - \tfrac{1}{2} R g_{ik} = T_{ik}.$$

c.f. E/dynamics $j_i = \delta \mathcal{L} / \delta A^i$

But in E/dynamics $\qquad L = \mathcal{L}_{matt} + F_{ij} F^{ij} + j_i A^i$
$\downarrow$ coupling term betw charges & field.

In grav. case no coupling term inserted. "Free-field" term only.

Machian. Cnst of $L_{mat} \rightarrow$ ideas of energy & mtm. $\therefore$ no need for coupling term. The coupling is automatic.
Varn. w.r.t. matter variables $\rightarrow$ material field eqns & varn. w.r.t. g^{ij} generates the energy-mtm. tensor as source of gravity. Anything with its own field $\rightarrow$ through $\delta \mathcal{L}/\delta g^{ij}$ a coupling to gravity.

Contracted Bianchi Identities needed to ensure cnse. of T^{ij}
We don't want to indepr. ens. for g^{ij}.
$\therefore$ if we insert cnse. we should be able to derive the C.B.I.

Can't produce full Bianchi identities ∵ these contain more than curve.
Whereas before we varied g^{ij} arbitrarily now we generate the var^n by a co-ordinate transf^n. Now the scalarity of L is invoked.

Th^m. The var^l deriv. w.r.t. g_{ik} of any scalar density depending only on g_{ik} & its derivs. up to any finite order has vanishing cov. deriv.

Pf.
Let it be $\mathcal{A}$. Put $\mathcal{A}^{ik} = \delta\mathcal{A}/\delta g_{ik}$ Sym. tensor density.
Consider $I = \int \mathcal{A}\, d\tau$

$$\delta I = \int \mathcal{A}^{ik}\, \delta g_{ik}\, d\tau$$

$\delta I = 0$ not now a physical hypothesis → field eqns which might be wrong.
Now $\delta I \equiv 0$ ∵ of scalarity → an identity now, not possibly incorrect eqns!

Let $x'^l = x^l + \lambda \phi^l(x^i) + \lambda^2 \psi^l(x^i) + \dots$ λ small.
Assume $x'^l = x^l$ on a bdry somewhere. Lts. of $\int$n. then unaffected.

δg_{ik} is local variation, viz. $g'_{ik}(x^s) - g_{ik}(x^s)$
But rule for tensor transf → $g'_{ik}(x'^s)$ → $\delta g_{ik} = \lambda\left(g_{lk}\frac{\partial\phi^l}{\partial x^i} + g_{im}\frac{\partial\phi^m}{\partial x^k} + \phi^n \frac{\partial g_{ik}}{\partial x^n}\right) + O(\lambda^2)$

Tensor transf^n. Gets us back to x.

Then $\delta I = \lambda \int \mathcal{A}^{ik}\left(\qquad\qquad\right) d\tau \equiv 0.$

Partially $\int$ → shift deriv. from ϕ to $\mathcal{A}$

$$= \int\left[-\frac{\partial \mathcal{A}^i{}_l}{\partial x^i}\phi^l - \frac{\partial \mathcal{A}^k{}_m}{\partial x^k}\phi^m + \mathcal{A}^{ik}\frac{\partial g_{ik}}{\partial x^n}\phi^n\right] d\tau = 0.$$

$$\therefore \int\left(-2\frac{\partial \mathcal{A}^k{}_m}{\partial x^k} + \mathcal{A}^{ik}\frac{\partial g_{ik}}{\partial x^m}\right)\phi^m\, d\tau \equiv 0.$$

But ϕ^m arb ∵ it generates co-ord. transf^n → required result.

$$\frac{\partial \mathcal{A}^k{}_m}{\partial x^k} - \tfrac{1}{2}\mathcal{A}^{ik}\frac{\partial g_{ik}}{\partial x^m} \equiv 0.$$

As $\mathcal{A}^{ik}_{\ \ v} = 0$, $\mathcal{A}^{km}{}_{;k} \equiv 0.$

Applications of th^m

i) $\mathcal{A} = \sqrt{g}$. $\mathfrak{g}^{ik}{}_{;k} = 0.$ Trivial ∵ $\mathfrak{g}^{ik}{}_{;l} = 0.$

ii) $\mathcal{A} = \sqrt{-g}R$ $\frac{\delta\sqrt{-g}R}{\delta g} = \sqrt{-g}\,(R^{ik} - \tfrac{1}{2}Rg^{ik})$

$$\left[\sqrt{-g}\,(R^{ik} - \tfrac{1}{2}Rg^{ik})\right]_{;i} = 0 \rightarrow (R^{ik} - \tfrac{1}{2}Rg^{ik})_{;i} = 0.$$

This in itself suggests the field eqns, viz $\Box T = 0$. satisfied.

Further identities.

Replace $\sqrt{-g}R$ by $\overset{=\Gamma}{\text{non-invariant}}$ Lagrangian density, differing only by a ∂gce., containing only 1st derivatives.

Consider $\mathcal{G}_{ik} = \Gamma^{\beta}_{\alpha k}\Gamma^{\alpha}_{i\beta} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{ik}$ $\mathcal{G} = \mathfrak{g}^{ik}\mathcal{G}_{ik}$.

$\sqrt{-g}R = -\mathcal{G}$ + ordinary ∂gce.
↓ no part in action.

Then $R_{ij} - \frac{1}{2}Rg_{ij} = -\delta\mathcal{G}/\delta\mathfrak{g}^{ij}$ This → field eqns.

__Also__ if $\delta\int\mathcal{G}\,d\tau = 0$ for real coord transf

$$(\mathfrak{T}^i_r + t^i_r)_{;i} = 0 \qquad t^i_r = \tfrac{1}{2}(-\mathcal{G}\delta^i_r + \mathcal{G}^i_{ik}\,\mathfrak{g}^{ik}{}_{,r})$$

where $-\mathcal{G}^r_{ik} = \dfrac{\partial\mathcal{G}}{\partial\mathfrak{g}^{ik}{}_{,r}}$

This is like en-mm. tensor of grav'l field. It __isn't__ a tensor density. $\mathcal{G}$ isn't a scalar density.
This is called the pseudo-energy tensor of grav'l field.
Can be made to vanish at a pt. in a frame (unlike a tensor). Principle of Equiv!

Definition is analogous to ord. class. field th? Canonical energy tensor.