DEPARTMENT OF PHYSICS
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Nov 20/80

Dear Alan :

Enclosed please find a few more notes. This discussion sets up a single algorithm for the construction of a multi-component jet. No iteration is required for the state variables, and the numerical work should present no problems (the eqns being only superficially complicated). Also enclosed is a little episode on Macsyma which reveals an error in previous notes.

Best wishes

Kayll

Streamline Relativistic Hydrodynamics of Jets

The problem:

If you assume that radio jets represent pressure confined outflows of matter, determine, under well defined but minimal assumptions, the confining pressure.

Solution:

First assume that the magnetic contribution to the confinement is negligible.

a) Background Hydrodynamics

Hydrodynamics is (given $T^{\alpha}_{\ \beta}$)

$$\nabla_\alpha n u^\alpha = 0, \tag{1}$$

with

$$\nabla_\alpha T^{\alpha}_{\ \beta} = 0. \tag{2}$$

Assume a perfect fluid,

$$T^{\alpha}_{\ \beta} = n h u^\alpha u_\beta + p\,\delta^{\alpha}_{\ \beta}, \tag{3}$$

where $h \equiv (\rho + p)/n$.

From (1)

$$(\ln n)^{\circ} + u = 0. \qquad (4)$$

With (1) through (4)

$$(h u_\beta)^{\circ} = -\frac{1}{n}\frac{\partial p}{\partial x^\beta} + \Gamma^{\gamma}_{\beta\alpha}\, h u^\alpha u_\gamma. \qquad (5)$$

Ignore the gravitational field and use cylindrical coordinates [Minkowski space: $(r, z, \varphi, t) \ni \overset{\circ}{t} = \gamma$].

Assume that $\forall$ scalar Φ

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial \varphi} = 0. \qquad (6)$$

Cylindrical Symmetry.

Stationary Flow.

With (6), (5) reads as

$$(h\gamma)^{\circ} = (h u_\varphi)^{\circ} = 0, \qquad (7)$$

$$(h u_r)^{\circ} = u_\varphi^2 \frac{h}{r^3} - \frac{1}{n}\frac{\partial p}{\partial r}, \qquad (8)$$

$$(h u_z)^{\circ} = -\frac{1}{n}\frac{\partial p}{\partial z}. \qquad (9)$$

b) Restricted Velocity Field

We assume $\exists\, g = g(r) \ni$

$$\frac{\partial}{\partial r}\left(g(r)\, \frac{u^r}{u^z} \right) = 0 . \qquad (10)$$

Then (4) reads as

$$n r u^z = A\, g(r) \ni \dot{A} = 0 . \qquad (11)$$

c) Equation of Motion

From (7)

$$h \gamma = B , \quad \dot{B} = 0 , \qquad (12)$$

and

$$u_\varphi = \gamma \Psi, \quad \dot{\Psi} = 0 . \qquad (13)$$

With (13),

$$\gamma^2 = \frac{r^2}{r^2 - \Psi^2} \left\{ 1 + u^{z2} \left(1 + \frac{u^{r2}}{u^{z2}} \right) \right\}, \qquad (14)$$

so that from the Bernoulli Eqn. (12)

$$u^{z2} = \frac{B^2 (r^2 - \Psi^2) - h^2 r^2}{h^2 r^2 \left\{ 1 + u^{r2}/u^{z2} \right\}}$$

$$= \frac{A^2 g^2(r)}{n^2 r^2}$$

from (11) for a non-axial streamline. Thus

$$A^2 \left(\frac{h}{n} \right)^2 g^2(r) \left\{ 1 + \left(\frac{u^r}{u^z} \right)^2 \right\} = r^2 \left\{ B^2 (1 - \Psi^2/r^2) - h^2 \right\} . \qquad (15)$$

d) Dimensionless Equation

Set $\Psi = 0$ ($\dot{\varphi} = 0$) here. Consider a fiducial point $\hat{o}$ along the streamline. Define

$$
\begin{aligned}
r/r_o &\equiv y \\
z/r_o &\equiv x + z_o/r_o \\
n/n_o &\equiv N(x) \\
h/h_o &\equiv H(x) \\
g/g_o &\equiv G = G(y, r_o) \\
1 + \left(\frac{dy}{dx}\right)_o^2 &\equiv \alpha
\end{aligned}
$$

The streamline defines $y(x)$, N & H are streamline functions.

Then, (15) reduces to

$$\left(\frac{dy}{dx}\right)^2 - f y^2 + 1 = 0 \tag{16}$$

$$\ni \quad f \equiv \left(\frac{N}{H}\right)^2 \frac{\alpha(\gamma_o^2 - H^2)}{G^2(\gamma_o^2 - 1)} . \tag{17}$$

Clearly $f > 0$ for real solutions y. Solutions to (16) are not unique: e.g. if $f = \beta = \text{const.}$

$y = 1/\sqrt{\beta}$ & $y = \frac{1}{\sqrt{\beta}} \cosh \sqrt{\beta}\, x$.

Why? — The Euler Eqns. (8) & (9) have not been used!

$\Rightarrow$ Invert (16) & (17):

$$\frac{1}{\alpha}\left(\frac{G}{y}\right)^2\left(\left(\frac{dy}{dx}\right)^2 + 1\right) = \left(\frac{N}{H}\right)^2 \frac{(\gamma_o^2 - H^2)}{(\gamma_o^2 - 1)}, \tag{18}$$

take $y(x)$ as given (observed).

This demands that the observed y can be taken as a streamline.

e) Equation of State

We suppose that the gas is an ideal neutral mix of protons & electrons.

i) N:
$$N = \sqrt{\frac{\mathcal{L}(\chi_e^0)}{\mathcal{L}(\chi_e)}\,\frac{\mathcal{L}(\chi_p^0)}{\mathcal{L}(\chi_p)}}$$

$$\ni \mathcal{L} \equiv \frac{\chi}{K_2(\chi)} \exp\left\{-\chi \frac{K_3(\chi)}{K_2(\chi)}\right\}, \quad \chi = \frac{mc^2}{kT}$$

$$\mathcal{L} \sim \begin{cases} \sqrt{\frac{2}{\pi}}\,\chi^{3/2}, & \chi >> 1 \quad (\text{Cold}) \\ \frac{1}{2}\chi^3 e^{-4}, & \chi << 1 \quad (\text{Hot}) \end{cases}$$

$$\therefore N \sim \begin{cases} (T/T_0)^{3/2}; & kT << m_e c^2 \quad ^{\dagger} \\ (T/T_0)^{9/4}; & m_e c^2 << kT << m_p c^2 \quad ^{\#} \end{cases}$$

† $T << 6\cdot 10^9\,K$, $^{\#}$ $6\cdot 10^9\,K << T << 10^{13}\,K$.

ii) h:
$$h = \frac{\sum_i n_i m_i G(\chi_i)}{\sum_j n_j}$$

$m \equiv mc^2$

$$\ni G(\chi) \equiv \frac{K_3(\chi)}{K_2(\chi)} \sim \begin{cases} 1 + 5/2\chi, & \chi >> 1 \quad (\text{Cold}) \\ 4/\chi, & \chi << 1 \quad (\text{Hot}) \end{cases}$$

$$\therefore h \sim \begin{cases} \frac{m_e + m_p}{2} \sim \frac{m_p}{2}; & kT << m_e c^2 \\ 2kT + \frac{m_p}{2} \sim \frac{m_p}{2}; & m_e c^2 << kT << m_p c^2 \end{cases}$$

$$\therefore H \sim 1$$

$$\therefore \frac{\gamma_0^2 - H^2}{\gamma_0^2 - 1} \sim 1$$

iii) p:
$$p = \sum_i n_i kT$$

$$\Rightarrow \mathcal{P} = N\left(\frac{T}{T_0}\right) \ni \mathcal{P} \equiv \frac{p}{p_0}$$

$$\therefore \left(\frac{N}{H}\right)^2\left(\frac{\gamma_0^2-H^2}{\gamma_0^2-1}\right) \simeq \begin{cases} P^{6/5} & ; \ kT \ll m_e c^2 \\ P^{18/13} & ; \ m_e c^2 \ll kT \ll m_p c^2 \end{cases} \tag{19}$$

From (18) & (19) then

$$\frac{1}{\alpha}\left(\frac{G}{y}\right)^2\left(\left(\frac{dy}{dx}\right)^2+1\right) \simeq \begin{cases} P^{6/5} & ; \ kT \ll m_e c^2 \\ P^{18/13} & ; \ m_e c^2 \ll kT \ll m_p c^2. \end{cases} \tag{20}$$

From (12)

$$H\gamma = \gamma_0 \tag{21}$$

so that $\gamma \sim$ const. for $H \sim 1$.

f) $\underset{\sim}{P}$ The analysis is restricted to a streamline "boundary". The boundary ought to be a boundary surface:

$$(T^\alpha_\beta n_\alpha n^\beta)^- = (T^\alpha_\beta n_\alpha n^\beta)^+ . \tag{22}$$

whereas $T^{\alpha\,-}_\beta = nhu^\alpha u_\beta + p\delta^\alpha_\beta$,

write $T^{\alpha\,+}_\beta = \bar{n}\bar{h}\, U^\alpha U_\beta + \bar{p}\,\delta^\alpha_\beta$

$\ni U^\alpha = (0,0,0,1)$ (Static cloud).

Then (22) $\Rightarrow$ $p = \bar{p} + \bar{n}\bar{h}\,(U^\alpha n_\alpha)^2$.

But $U^\alpha n_\alpha = n_t = 0$ (Static flow)

$$\therefore \quad P = \bar{P},$$

i.e. P gives the dimensionless pressure of the external cloud (i.e. in the rest frame of the cloud).

g) <u>Procedure</u> From (20) then

<u>assumed velocity field</u>

$$P = \left\{ \frac{1}{\alpha} \left(\frac{G}{y} \right)^2 \left(\left(\frac{dy}{dx} \right)^2 + 1 \right) \right\}^{\delta} \ni \delta = \begin{cases} 5/6 \ ; & kT \ll m_e c^2 \\ 13/18 \ ; & m_e c^2 \ll kT \ll m_p c^2 \end{cases}$$

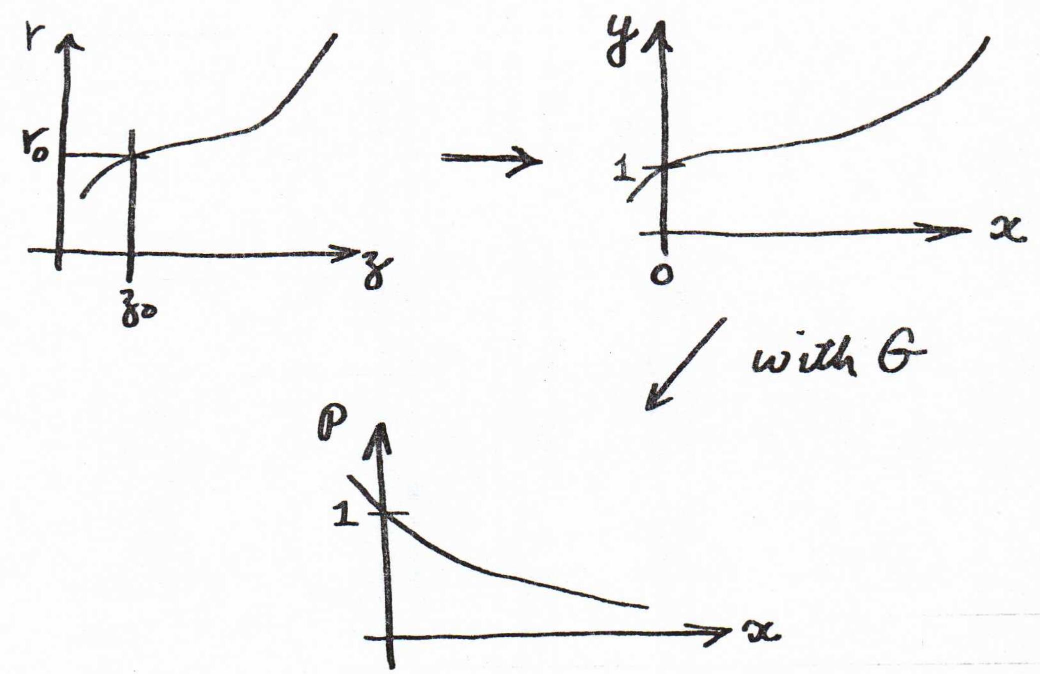
external Cloud.

<u>Observed</u>

<u>The "parameter space" is simply G</u>.

Need α, r_0, z_0.

Take an <u>observed</u> part of the jet as the fiducial point.



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Nov 12/80

Dear Alan:

If what I enclose is correct, Dick's streamline approach (when corrected) and my approach finally give the same equations. The extra term indicates that a thermal component will change d^2R/dz^2 in, perhaps, an important way, and hence the shape of the boundary $R(z)$. As the 'bulk' energy becomes relativistic, the term probably does not contribute much. However, a thermal component will change even a 'fast' jet.

Kayll

Refer to the notes: Relativistic non-Magnetic Steady Jets Ⓐ:

- p.5: The continuity eqn. actually reads $\{R^2 \omega_z n\}^* = 0$ so $n \neq n(z)$ in general.

- p.8: From the $\omega_r\, d\omega_r/dz|_B$ & $\omega_z\, d\omega_z/dz|_B$ eqns. of the Summary we have:

$$(*)\quad \omega_z^2 \frac{d^2R}{dz^2} = \frac{\gamma_B^2 \psi_S^2}{R^3} + \frac{2}{\delta_B R}(p_i - p_E)\left(1 + \left(\frac{dR}{dz}\right)^2\right) + \frac{1}{\delta_B}\left(\frac{dR}{dz}\right)\frac{dp_E}{dz}.$$

<u>New Summary for u-similarity</u>:

Continuity : $\{R^2\omega_z n\}^* = 0$ so $R^2 \omega_z n = f(r/R)$
Bernoulli : $\{\xi\gamma\}^* = 0$ so $\xi\gamma = \delta\gamma/n = g(r/R)$
Boundary : $dR/dz = \omega_r/\omega_z$
Rotation : $\omega_\varphi = \gamma\psi/R \ni \psi^* = 0 \quad (\psi_B = \psi_S)$
Euler : $(*)$ given above.

Here $^* \equiv u_r\, \partial/\partial r + u_z\, \partial/\partial z$, $\gamma_B^2 = 1 + \omega^2/c^2$ ($\simeq 1 + \omega_z^2/c^2$), and $\delta_B = p_i + p_E = \rho_B + p_B$ (in appropriate units, <u>see below</u>).

Compare Dick's Eqns. (Oct 30/80 p. 14). i) $nh\ (\equiv \delta)$ is $nh|_B$. ii) To get his $(*)$, <u>linearize</u> (drop $(dR/dz)^2$ and dp_E/dz terms) $(*)$ here.

<u>Questions</u>: i) Is it in fact much more difficult to solve (numerically) the unlinearized form $(*)$?

②

ii) Are there no cases when the dp_E/dz term will be of interest? (It may be that we can make some interesting jets with this term included.)

iii) Use $dR/dz = \omega_r/\omega_z$ and $\omega_\varphi = \gamma\Psi/R$ in the exact form for γ_B^2 to re-write (*) in the form:

$$(*') \quad \omega_z^2 \frac{d^2R}{dz^2} = \frac{\Psi_S^2 c^2}{R(c^2R^2-\Psi_S^2)} + \frac{1}{R}\left(1+\left(\frac{dR}{dz}\right)^2\right)\left(\frac{2(p_j-p_E)}{\delta_B} + \frac{\Psi_S^2\omega_z^2}{c^2R^2-\Psi_S^2}\right) + \frac{1}{\delta_B}\left(\frac{dR}{dz}\right)\frac{dp_E}{dz}$$

Though $\gamma_B^2 \simeq 1+\omega_z^2/c^2$ can be an excellent approximation, is it in fact more difficult to handle (numerically) the above exact eqn.? If $c^2R_S^2 \gg \Psi_S^2$ then, with accuracy guaranteed we can use Ψ_S^2/R^3 for the first term and $(\Psi_S/R)^2(\omega_z/c)^2$ for the second term in the second bracket. The "relativistic" terms are then just the $(\omega_z/c)^2$ term and the form of δ_B, in (*').

Change units of Ⓐ why?

Units: We use std. (but relativistic) units. Take $[p] = [\delta]$ (see below) so replace δ above by δ/c^2 in these eqns. i.e. $[\Psi] = L^2/t$, $[(*')] = [(*)] = L/t^2$, with the $/c^2$ added to δ and $[p]$ taken $= [\delta]$.

Eqn. of State: Take the std. relativistic Maxwell-Boltzmann gas, and choose T as the "fundamental" variable (i.e. n, ρ, p as fns of T)

$n \equiv$ # baryons/unit vol. in local rest frame; $\rho \doteq T^{\alpha\beta}u_\alpha u_\beta$

$p \equiv T^{\alpha\beta}n_\alpha n_\beta = \frac{1}{3}T^{\alpha\beta}(u_\alpha u_\beta + \eta_{\alpha\beta})$.

$[p] = [\rho]$ $\quad p = nkT$,

$ML^{-1}t^{-2}$. $\quad \rho = p(3 + x K_1(x)/K_2(x)) \ni x \equiv mc^2/kT$, single comp. rel. gas cont.

$$\therefore\ \mathcal{E} \equiv \rho + p = p(4 + \chi K_1/K_2), \quad [\mathcal{E}] = [p].$$

$[n]$ L^{-3} $$n = b\,T K_2 \exp(\chi K_1/K_2) \ni b = \exp(4 - s/k)\cdot 4\pi m^2 c\, k/h^3$$

$s \equiv$ specific entropy (const. here as in all perf. fluids.)

(Note: $[R^2 \omega_z n] = t^{-1}$, $[\mathcal{E}\gamma/nc^2] = M$.)

$$\left.\begin{aligned}
\therefore\ p &= (bk)T^2 K_2 \exp(\chi K_1/K_2) \\
\mathcal{E} &= (bk)T^2(4K_2 + \chi K_1)\exp(\chi K_1/K_2) \\
\mathcal{E}/n &= kT(4 + \chi K_1/K_2)
\end{aligned}\right\}\ \text{single comp. Eqn. of State}$$

$m = $ const., $T = T(z)$, $K_n = K_n(\chi)$, $\chi \equiv mc^2/kT$.

<u>Continuity Eqn. reads:</u>

$$\left(\frac{R}{R_s}\right)^2\left(\frac{\omega_z}{\omega_{zs}}\right)\left(\frac{T}{T_s}\right)\left(\frac{K_2}{K_{2s}}\right)\exp(\chi K_1/K_2 - \chi_s K_{1s}/K_{2s}) = 1.$$

<u>Bernoulli Eqn. reads:</u>

$$\left(\frac{\gamma}{\gamma_s}\right)_B\left(\frac{T}{T_s}\right)\left(\frac{4 + \chi K_1/K_2}{4 + \chi_s K_{1s}/K_{2s}}\right) = 1,$$

$$\ni \left(\frac{\gamma}{\gamma_s}\right)_B = \left(\frac{R}{R_s}\right)\left(\frac{R_s^2 c^2 - \psi_s^2}{R^2 c^2 - \psi_s^2}\cdot\frac{1 + (\omega_z/c)^2(1 + (dR/dz)^2)}{1 + (\omega_{zs}/c)^2(1 + (dR/dz)_s^2)}\right)^{1/2}$$

$$= \left(\frac{1 + (\omega_z/c)^2(1 + (dR/dz)^2)}{1 + (\omega_{zs}/c)^2(1 + (dR/dz)_s^2)}\right)^{1/2} \text{ for } R_s^2 c^2 \gg \psi_s^2.$$

<u>For the Euler Eqn:</u>

<u>Pressure gradient</u>

$$\frac{c^2}{\mathcal{E}_B}\left(\frac{dR}{dz}\right)\frac{dp_E}{dz} = \frac{c^2}{4 + \chi K_1/K_2}\left(\frac{dR}{dz}\right)\frac{d\ln p_E}{dz},$$

$$\frac{d\ln p_E}{dz} = \frac{1}{T}\frac{dT}{dz}\left\{4 - \chi\left(\frac{3K_1}{K_2} + \chi\left(\left(\frac{K_1}{K_2}\right)^2 - 1\right)\right)\right\}.$$

$$-\frac{c^2 p_E}{\delta_B} = -\frac{c^2 K_2}{4K_2 + \chi K_1} = -\frac{c^2}{4 + \chi K_1/K_2}$$

To complete Euler Eqn. need p_i. Apply Bernoulli along axis

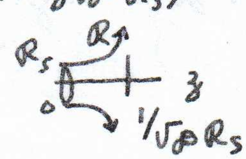
$$\left(\frac{1+(\omega_z/c)^2}{1+(\omega_{zs}/c)^2}\right)\left[\left(\frac{T}{T_S}\right)\left(\frac{4+\chi K_1/K_2}{4+\chi_S K_{1S}/K_{2S}}\right)\right]_i = 1$$

Aside: Suppose $T = T_S$ (isothermal cloud). From continuity $(R/R_S)^2 = \omega_{zs}/\omega_z$. From Bernoulli $(\gamma/\gamma_s)_B = 1$ so if $R_S^2 c^2 >> \psi_s^2$

$$1 + (dR/dz)^2 = (\omega_{zs}/\omega_z)^2 (1 + (dR/dz)_s^2)$$
$$= \frac{R^4}{R_S^4}(1 + (dR/dz)^2_s)$$

$\therefore\ 1 + (dR/dz)^2 = \mathcal{D} R^4$, $\mathcal{D}$ const.

$\therefore\ \displaystyle\int_s^z \frac{dR}{\sqrt{\mathcal{D}R^4 - 1}} = z$, $\mathcal{D}R^4 >> 1 \approx R = R_s/(1 - z\sqrt{\mathcal{D}}R_s)$



From Euler $(\partial\psi_s = 0)$

$$\omega_z^2 d^2R/dz^2 = \mathcal{E} R^3 (p_i - p_E), \quad \mathcal{E} \text{ const}$$

so $d^2R/dz^2 = \mathcal{F} R^7 (p_i - p_E)$. But with Bernoulli & $\mathcal{D}R^4 >> 1$, $d^2R/dz^2 = \mathcal{G} R$ so $(p_i - p_E) \propto 1/R^6$.

Ultra-Relativistic Energies : $kT >> mc^2$

$$\left.\begin{aligned} K_2 &\sim 2/\chi^2 \sim 2k^2T^2/m^2c^4 \\ K_1 &\sim 1/\chi \sim kT/mc^2 \end{aligned}\right\} \begin{aligned} &\chi K_1/K_2 \sim 0 \\ &\chi(3K_1/K_2 + \chi((K_1/K_2)^2 - 1)) \sim 0 \end{aligned}$$

Continuity $\rightarrow$ $\left(\frac{R}{R_S}\right)^2\left(\frac{\omega_z}{\omega_{zs}}\right)\left(\frac{T}{T_S}\right)^3 = 1$

Bernoulli $\rightarrow$ $\left(\frac{\gamma}{\gamma_s}\right)_B\left(\frac{T}{T_S}\right) = 1$

$$\frac{c^2}{\delta_B}\left(\frac{dR}{dz}\right)\frac{dp_E}{dz} \sim \frac{c^2}{T}\frac{dT}{dz}\frac{dR}{dz}, \quad -\frac{c^2 p_E}{\delta_B} \sim -\frac{c^2}{4},$$

$$\text{axis} \to \quad \frac{1+(\omega_z/c)^2}{1+(\omega_{zs}/c)^2}\left(\frac{T_s}{T_{ss}}\right) = 1.$$

<u>Classical Energies:</u> $kT \ll mc^2$

$$K_1 \sim K_2 \sim \sqrt{\frac{\pi}{2x}}\, e^{-x}, \quad xK_1/K_2 \sim x$$

$$x(3K_1/K_2 + x(K_1^2/K_2^2 - 1)) \sim 3x.$$

$$\text{Continuity} \to \left(\frac{R}{R_s}\right)^2\left(\frac{\omega_z}{\omega_{zs}}\right)\left(\frac{T}{T_s}\right)^{3/2} = 1$$

$$\text{Bernoulli} \to \left(\frac{\gamma}{\gamma_s}\right)_B\left(\frac{4kT + mc^2}{4kT_s + mc^2}\right) = 1 \simeq \left(\frac{\gamma}{\gamma_s}\right)_B.$$

$$\frac{c^2}{\delta_B}\left(\frac{dR}{dz}\right)\frac{dp_E}{dz} \to \frac{c^2}{4kT+mc^2}\left(\frac{dR}{dz}\right)\frac{1}{T}\frac{dT}{dz}\left\{4kT - 3mc^2\right\}$$

$$\simeq -\frac{3c^2}{T}\frac{dT}{dz}\left(\frac{dR}{dz}\right)$$

$$-\frac{c^2 p_E}{\delta_B} \to -\frac{c^2 kT}{4kT+mc^2} \simeq -\frac{kT}{m},$$

$$\text{axis} \to \left(\frac{1+(\omega_z/c)^2}{1+(\omega_{zs}/c)^2}\right)\left[\frac{4kT+mc^2}{4kT_s+mc^2}\right]_{s} = 1 \simeq \frac{1+(\omega_z/c)^2}{1+(\omega_{zs}/c)^2}$$

$$2p_s c^2/\delta_B \to \frac{2kc^2 T_s^{5/2}}{T^{3/2}(4kT+mc^2)} \simeq \frac{2kT_s^{5/2}}{m\,T^{3/2}}$$

[Classical theory has <u>low energy</u> & <u>low vel.</u> so

$$\omega_z^2 \frac{d^2R}{dz^2} = \frac{\psi_s^2}{R^3} + \frac{1}{R}\left(1+\left(\frac{dR}{dz}\right)^2\right)\left(\frac{2k}{mT^{3/2}}\right)\left(T_s^{5/2} - T^{5/2}\right) - \frac{3c^2}{T}\frac{dT}{dz}\frac{dR}{dz}.$$

Multi-component gas: (i species)

Specify m_i, T_i

then $p = \Sigma_i p_i$, $\rho = \Sigma_i \rho_i$, $n = \Sigma_i n_i$, $\delta = \Sigma_i \delta_i$.

Can proceed as with $i = 1$ (single component) but sum

For a thermal ($\hat{T}$) and relativistic ($\hat{R}$) 2-component model, specify m_T & m_R (fixed). One additional free function must be specified, e.g. $T_T / T_R \equiv \tau$ ($= \tau(r/R)$ if $\tau^* = 0$).

Ⓕ March 4

Arno's Procedure

① Specify an equation of state along the jet axis (ζ). Use Bernoulli along the axis so

$$\left.\frac{\gamma}{\gamma_s}\right|_\zeta = \left.\frac{n}{n_s}\,\frac{\delta_s}{\delta}\right|_\zeta$$

$$\Rightarrow \omega_z(z)/\omega_z^s .$$

② Use Continuity along the boundary (B)

$$\frac{R}{R_s} = \sqrt{\frac{n_s}{n_B}\,\frac{\omega_z^s}{\omega_z(z)}}$$

IF we know n_B then R is known. But n_B must be $n_B(z)$, which in turn must be n_ζ, which we have specified in ①.

Thus R is already known.

Further Notes

Though $p = p(z)$ along a streamline, across the jet $p \neq p(z)$, it is governed by the homology. But $p = n\,kT$ (e.g. 1 component) & $n = n(z)$, $\therefore$ $T \neq T(z)$ across the jet, T is also governed by the homology. Thus the jet is already 2-dimensional in T, and with $p_j > p_B$, $T_B < T_s$. This is extended to a mixture as in notes Ⓔ, for this procedure apply Ⓔ along the axis (ideal gas, $T \neq 0$)

first use classical eqns?

We still need Euler: Suppose $\psi = 0$ (no rotation), then

$$\omega_z^2 \frac{d^2R}{dz^2} = \frac{2c^2(p_j - p_B)}{\delta_B R}\left(1 + \left(\frac{dR}{dz}\right)^2\right) + \frac{c^2}{\delta_B}\left(\frac{dR}{dz}\right)\frac{dp_E}{dz}.$$

algebraic (Since we know R, we find δ_B from Bernoulli along B.)

Euler now gives p_E. The jet constructed needs a background, and Euler gives the required one.

$$p_E \simeq -\delta_B R\omega_z^2\, d^2R/dz^2 / 2c^2(1+(dR/dz)^2) + p_j.$$

Given the jet (axial construction as above) & given a background, you must adjust ψ for a consistent Euler eqn. (Euler is now at worst 1st order)

- ρ_i is, in general $\neq \rho_B$ as written previous in notes
- drop ③ & following in Ⓑ.

Relativistic homologous jets : – non magnetic, non-rotating – { one comp.; mix.

– magnetic, "

– rot ?

Relativistic Non-Magnetic Steady Jets

Notation: $T^{\alpha\beta} = (\rho + p/c^2)\, u^\alpha u^\beta + p\, \eta^{\alpha\beta}$

$u^\alpha = \gamma(1, v^i)$; $\gamma = 1/\sqrt{1 - v^2/c^2}$; $u^\alpha u_\alpha = -1.$

Euler Eqn: $$\nabla\left(\frac{v^2}{2}\right) + \frac{1}{\gamma^2(\rho + p/c^2)} \nabla p = v \times (\nabla \times v) \quad ①$$

Continuity: $$\gamma\, v \cdot \nabla \rho + (\rho + p/c^2)\, \nabla \gamma v = 0 \quad ②$$

(The Bernoulli Eqn. $v \cdot ① = 0$ not used here.)

Assume axial symmetry (no φ dept. terms), $\overset{*}{} \equiv v_r \frac{\partial}{\partial r} + v_z \frac{\partial}{\partial z}$

then the components of ① read

$$\overset{*}{v}_r = \frac{v_\varphi^2}{r} - \frac{1}{\gamma^2(\rho + p/c^2)} \frac{\partial p}{\partial r}, \quad (1a)$$

$$\overset{*}{v}_\varphi = -\frac{v_r v_\varphi}{r} \Rightarrow v_\varphi = \frac{L}{r} \ni \overset{*}{L} = 0, \quad (1b)$$

$$\overset{*}{v}_z = -\frac{1}{\gamma^2(\rho + p/c^2)} \frac{\partial p}{\partial z}. \quad (1c)$$

Continuity reads

$$\gamma \overset{*}{\rho} = -(\rho + p/c^2) \left\{ \overset{*}{\gamma} + \gamma \left(\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} \right) \right\}. \quad (2a)$$

Similarity Forms:

$$p = p_s - (p_s - p_R) \left(\frac{r}{R}\right)^2$$

$$L = \Omega R_A^2 \left(\frac{r}{R}\right)^2$$

$$v = \left(\omega_r \left(\frac{r}{R}\right), \omega_\varphi \left(\frac{r}{R}\right), \omega_z\right)$$

Along the boundary streamline then:

$$\frac{dR}{dz} = \frac{\omega_r}{\omega_z},$$

$$\omega_z \frac{d\omega_r}{dz} = \frac{\Omega^2 R_A^4}{R^3} + \frac{2(p_s - p_R)}{\gamma_R^2(\rho + p_R/c^2)R}, \qquad (1a')$$

$$\omega_\varphi = \frac{\Omega R_A^2}{R} \qquad (1b')$$

$$\omega_z \frac{d\omega_z}{dz} = \frac{-1}{\gamma_R^2(\rho + p_R/c^2)} \left\{ \frac{dp_R}{dz} + 2\frac{\omega_r}{\omega_z} \cdot \frac{(p_s - p_R)}{R} \right\}, \qquad (1c')$$

$$\omega_z \frac{d\rho}{dz} = -(\rho + p_R/c^2) \left\{ \frac{2\omega_r}{R} + \frac{d\omega_z}{dz} + \frac{\gamma_R^2 \omega_z}{2C^2} \frac{d\omega^2}{dz} \right\}, \qquad (2a')$$

where $p_R(z)$ given

$$\omega^2 \equiv \omega_r^2 + \omega_\varphi^2 + \omega_z^2$$

$$\gamma_R \equiv 1/\sqrt{1 - \omega^2/c^2}.$$

- Specify equation of state,
- to get non-relativistic case set all $\frac{\text{terms}}{c^2} = 0$.

Components of the Euler eqn. ⑤ read

$$\left.\begin{aligned}
\omega_r \frac{d\omega_r}{dz} &= \frac{dR}{dz}\left\{\frac{\psi^2}{R^3} - \left(\frac{R}{r}\right)\frac{1}{\gamma^2\delta}\frac{\partial p}{\partial r}\right\} \\
\omega_\varphi &= \frac{\psi}{R} \ni \psi^* = 0 \\
\omega_z \frac{d\omega_z}{dz} &= -\frac{1}{\gamma^2\delta}\frac{\partial p}{\partial z}
\end{aligned}\right\} \quad (7)$$

For the continuity eqn. ③ $\gamma\overset{*}{\rho} + \delta\overset{*}{\gamma} + \delta\gamma\,\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{2\omega_r}{R} + \frac{d\omega_z}{dz} = \omega_z\frac{d\ln R^2\omega_z}{dz}$$

$$\overset{*}{\rho} = \omega_r\left(\frac{r}{R}\right)\frac{\partial\rho}{\partial r} + \omega_z\frac{\partial\rho}{\partial z} = \omega_z\frac{d\rho}{dz} \qquad \text{for } \underline{\rho = \rho(z)}$$

$$\overset{*}{\gamma} = \frac{\gamma^3}{2c^2}\left\{\left(\frac{r}{R}\right)^2\left(\overset{*}{\omega}{}_r^2 + \overset{*}{\omega}{}_\varphi^2\right) + \overset{*}{\omega}{}_z^2\right\}$$

$$\overset{*}{\omega}{}_r^2 = 2\omega_r\omega_z\frac{d\omega_r}{dz} = 2\omega_z\frac{dR}{dz}\left\{\frac{\psi^2}{R^3} - \left(\frac{R}{r}\right)\frac{1}{\gamma^2\delta}\frac{\partial p}{\partial r}\right\}$$

$$\overset{*}{\omega}{}_\varphi^2 = \frac{-2\omega_\varphi^2\omega_r}{R} = -\frac{2\psi^2}{R^3}\omega_z\frac{dR}{dz}$$

$$\overset{*}{\omega}{}_z^2 = 2\omega_z^2\frac{d\omega_z}{dz} = -\frac{2\omega_z}{\gamma^2\delta}\frac{\partial p}{\partial z}$$

$$\therefore \overset{*}{\gamma} = -\frac{\gamma\omega_z}{c^2\delta}\left\{\left(\frac{r}{R}\right)\frac{\partial p}{\partial r}\frac{dR}{dz} + \frac{\partial p}{\partial z}\right\}$$

Thus the continuity eqn. reads

$$\delta\frac{d\ln R^2\omega_z}{dz} + \frac{\partial\rho}{\partial z} + \left(\frac{r}{R}\right)\frac{\partial\rho}{\partial r}\frac{dR}{dz} - \frac{1}{c^2}\left\{\left(\frac{r}{R}\right)\frac{\partial p}{\partial r}\frac{dR}{dz} + \frac{\partial p}{\partial z}\right\} = 0 \quad (8)$$

Again let $p = p_j - (p_j - p_E)\left(\frac{r}{R}\right)^2 \ni p = p(z)$ (9)

$$\therefore \begin{cases} \dfrac{\partial p}{\partial r} = \dfrac{2r}{R^2}(p_E - p_j) \\ \dfrac{\partial p}{\partial z} = \dfrac{dp_j}{dz} + \left(\dfrac{r}{R}\right)^2\left(\dfrac{dp_E}{dz} - \dfrac{dp_j}{dz}\right) - \dfrac{2}{R}\left(\dfrac{r}{R}\right)^2 \dfrac{dR}{dz}(p_E - p_j) \end{cases}$$

$$\Rightarrow \left(\frac{r}{R}\right)\frac{\partial p}{\partial r}\frac{dR}{dz} + \frac{\partial p}{\partial z} = \frac{dp_j}{dz} + \left(\frac{r}{R}\right)^2\left(\frac{dp_E}{dz} - \frac{dp_j}{dz}\right)$$

Summary: Under the v-similarity & (9) & $\rho = \rho(z)$:

$$\frac{dR}{dz} = \frac{\omega_r}{\omega_z}$$

$\gamma_B = 1/\sqrt{1-\omega^2/c^2}$

$$\omega_\varphi = \frac{\psi}{R}, \quad \psi^* = 0 \quad (\omega_\varphi \neq \omega_\varphi(z))$$

$\delta_B = \rho_j + p_E/c^2$

$\delta_j = \rho_j + p_j/c^2$

$$\left.\omega_r \frac{d\omega_r}{dz}\right|_B = \frac{dR}{dz}\left\{\frac{\psi^2}{R^3} + \frac{2(p_j - p_E)}{R\gamma_B^2 \delta_B}\right\}$$

$$\left.\omega_z \frac{d\omega_z}{dz}\right|_B = -\frac{1}{\gamma_B^2 \delta_B}\left\{\frac{dp_E}{dz} + \frac{2}{R}\frac{dR}{dz}(p_j - p_E)\right\}$$

Continuity

$$\delta \frac{d \ln R^2 \omega_z}{dz} + \frac{d\rho}{dz} - \frac{1}{c^2}\left\{\frac{dp_j}{dz} + \left(\frac{r}{R}\right)^2\left(\frac{dp_E}{dz} - \frac{dp_j}{dz}\right)\right\} = 0$$

on axis: $$\delta_j \frac{d \ln R^2 \omega_z}{dz} + \frac{d\rho}{dz} - \frac{1}{c^2}\frac{dp_j}{dz} = 0$$

on boundary: $$\delta_B \frac{d \ln R^2 \omega_z}{dz} + \frac{d\rho}{dz} - \frac{1}{c^2}\frac{dp_E}{dz} = 0$$

$$\therefore \left(\frac{p_s - p_E}{c^2}\right)\frac{d\ln R^2\omega_z}{dz} = \frac{d}{dz}\left(\frac{p_s - p_E}{c^2}\right)$$

$$\therefore \frac{p_s - p_E}{c^2} \backsim R^2\omega_z \qquad (10)$$

(10) is a relativistic result for v-similarity.

Classical result: drop $\frac{1}{c^2}$ terms so $R^2\omega_z\rho = \text{const.}$

4: u-similarity: $u_r = \omega_r\left(\frac{r}{R}\right)$, $u_\varphi = \omega_\varphi\left(\frac{r}{R}\right)$, $u_z = \omega_z$.

Continuity: from (1) $\underset{\sim}{\nabla}\cdot n\underset{\sim}{u} = n\underset{\sim}{\nabla}\cdot\underset{\sim}{u} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}n = 0$; $\underset{\sim}{u} \equiv \gamma\underset{\sim}{v}$

$$\therefore n\left\{\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right\} + u_r\frac{\partial n}{\partial r} + u_z\frac{\partial n}{\partial z} = 0.$$

$$\therefore n\left\{2\frac{\omega_r}{R} + \frac{d\omega_z}{dz}\right\} + \omega_z\frac{dn}{dz} = 0 \qquad \underline{\omega_{r,z} = \omega(z),\ n = n(z)}$$

$$\therefore n\omega_z\frac{d\ln R^2\omega_z}{dz} + \omega_z\frac{dn}{dz} = 0 \Rightarrow \boxed{R^2\omega_z n = \text{const.}} \qquad (11)$$

Euler: Since $(\xi\gamma)^* = 0$ from p. (2) we have

$$\left.\begin{aligned}
(\xi u_r)^* &= \frac{\xi\gamma v_\varphi^2}{r} - \frac{\xi}{\gamma\delta}\frac{\partial p}{\partial r} = \frac{\xi}{\gamma}\left(\frac{u_\varphi^2}{r} - \frac{1}{\delta}\frac{\partial p}{\partial r}\right)\\
(\xi u_\varphi)^* &= -\frac{\xi\gamma v_r v_\varphi}{r} = -\frac{\xi u_r u_\varphi}{\gamma r}\\
(\xi u_z)^* &= -\frac{\xi}{\gamma\delta}\frac{\partial p}{\partial z}
\end{aligned}\right\}$$

$$\therefore \quad \underline{\underline{r}} \qquad \omega_r \frac{d\omega_r}{dz} + \omega_r \gamma \frac{dR}{dz} (\ln \xi)^* = \frac{dR}{dz} \left\{ \frac{\omega_\varphi^2}{R} - \frac{R}{r} \frac{1}{\delta} \frac{\partial p}{\partial r} \right\}$$

$$\underline{\underline{z}} \qquad \omega_z \frac{d\omega_z}{dz} \gamma + \omega_z \gamma (\ln \xi)^* = -\frac{1}{\delta} \frac{\partial p}{\partial z}$$

$$\underline{\underline{\varphi}} \qquad v_\varphi = \frac{\mathcal{L}}{r} \ni \overset{*}{\mathcal{L}} = 0.$$

$$\frac{\omega_\varphi^2}{R} = \frac{\gamma^2 \psi^2}{R^3} \leftarrow \quad \omega_\varphi = \gamma \left(\frac{R}{r} \right) v_\varphi = \frac{\gamma R \mathcal{L}}{r^2} = \frac{\gamma \mathcal{L}}{R} \left(\frac{R}{r} \right)^2 = \frac{\gamma \psi}{R} \ni \overset{*}{\psi} =$$

$$\therefore \quad \overset{*}{\omega}_\varphi = \frac{\psi \overset{*}{\gamma}}{R} - \frac{\psi \gamma \overset{*}{R}}{R^2} = \frac{\psi}{R} \left(\overset{*}{\gamma} - \frac{\omega_r}{R} \right)$$

---

from Bernoulli Eqn $\quad (\ln \xi)^* = -(\ln \gamma)^*$

$$(\ln \gamma)^* = \frac{\overset{*}{\gamma}}{\gamma} = \frac{1}{2\gamma^2} \left\{ 1 + \frac{1}{c^2} \left(\left(\frac{r}{R} \right)^2 (\omega_r^2 + \omega_\varphi^2) + \omega_z^2 \right) \right\}^*$$

$$= \frac{1}{2c^2 \gamma^2} \left\{ \left(\frac{r}{R} \right)^2 \left(\overset{*}{\omega}{}_r^2 + \overset{*}{\omega}{}_\varphi^2 \right) + \overset{*}{\omega}{}_z^2 \right\}$$

$$\overset{*}{\omega}{}_r^2 = 2\omega_r \overset{*}{\omega}_r = \frac{2\omega_r \omega_z}{\gamma} \frac{d\omega_r}{dz}$$

$$= \frac{2\omega_z}{\gamma} \left\{ \frac{\omega_r}{\omega_z} \left\{ \frac{\gamma^2 \psi^2}{R^3} - \frac{R}{r\delta} \frac{\partial p}{\partial r} \right\} + \frac{\gamma \omega_r^2}{\omega_z} (\ln \gamma)^* \right\}$$

$$\overset{*}{\omega}{}_\varphi^2 = 2\omega_\varphi \overset{*}{\omega}_\varphi = \frac{2\gamma \psi^2}{R^2} \left(\overset{*}{\gamma} - \frac{\omega_r}{R} \right) = \frac{2\gamma^2 \psi^2}{R^2} \left((\ln \gamma)^* - \frac{\omega_r}{\gamma R} \right)$$

$$\overset{*}{\omega}{}_z^2 = 2\omega_z \overset{*}{\omega}_z = \frac{2\omega_z^2}{\gamma} \frac{d\omega_z}{dz} = \frac{2\omega_z}{\gamma} \left\{ -\frac{1}{\delta} \frac{\partial p}{\partial z} + \omega_z \gamma (\ln \gamma)^* \right\}$$

$$\therefore\ \overset{*}{\omega}_r^2 + \overset{*}{\omega}_\varphi^2 = \cancel{\frac{2\omega_r \gamma \psi^2}{R^3}} - \frac{2\omega_r R}{\gamma r \delta}\frac{\partial p}{\partial r} + 2\omega_r^2(\ln\gamma)^*$$

$$+ \frac{2\gamma^2\psi^2}{R^2}(\ln\gamma)^* - \cancel{\frac{2\gamma\omega_r\psi^2}{R^3}}$$

$$= 2(\ln\gamma)^*\left\{\omega_r^2 + \frac{\gamma^2\psi^2}{R^2}\right\} - \frac{2\omega_r}{\gamma\delta}\left(\frac{R}{r}\right)\frac{\partial p}{\partial r}$$

$$\left(\frac{r}{R}\right)^2\left(\overset{*}{\omega}_r^2 + \overset{*}{\omega}_\varphi^2\right) = 2(\ln\gamma)^*\left\{\left(\frac{r}{R}\right)^2\omega_r^2 + \frac{\gamma^2\psi^2 r^2}{R^4}\right\} - \frac{2\omega_r}{\gamma\delta}\left(\frac{r}{R}\right)\frac{\partial p}{\partial r}$$

$$\overset{*}{\omega}_z^2 = 2\omega_z^2(\ln\gamma)^* - \frac{2\omega_z}{\gamma\delta}\frac{\partial p}{\partial z}$$

$$(\ln\gamma)^* = \frac{1}{2c^2\gamma^2}\left\{2(\ln\gamma)^*\left\{\omega_z^2 + \left(\frac{r}{R}\right)^2\omega_r^2 + \frac{\gamma^2\psi^2 r^2}{R^4}\right\} - \frac{2}{\gamma\delta}\left\{\omega_z\frac{\partial p}{\partial z} + \omega_r\left(\frac{r}{R}\right)\frac{\partial p}{\partial r}\right\}\right\}.$$

$$\therefore\ (\ln\gamma)^* = -(\ln\xi)^*$$

$$= \frac{-\frac{\cancel{2}}{\gamma\delta}\left\{\omega_z\frac{\partial p}{\partial z} + \omega_r\left(\frac{r}{R}\right)\frac{\partial p}{\partial r}\right\}}{\cancel{2}\left\{c^2\gamma^2 - \omega_z^2 - \left(\frac{r}{R}\right)^2\omega_r^2 - \frac{\gamma^2\psi^2 r^2}{R^4}\right\}}$$

$$\therefore\ \gamma(\ln\xi)^* = -\overset{*}{\gamma}$$

$$= \frac{\omega_z\frac{\partial p}{\partial z} + \omega_r\left(\frac{r}{R}\right)\frac{\partial p}{\partial r}}{c^2\gamma^2\delta - \delta\left(\omega_z^2 + \left(\frac{r}{R}\right)^2\omega_r^2 + \frac{\gamma^2\psi^2 r^2}{R^4}\right)}$$

$$\therefore \gamma(\ln \xi)^* = -\overset{*}{\gamma} = \frac{\omega_z \left\{ \frac{\partial p}{\partial z} + \left(\frac{r}{R}\right) \frac{dR}{dz} \frac{\partial p}{\partial r} \right\}}{\delta \left(c^2\gamma^2 - \left(\omega_z^2 + \left(\frac{r}{R}\right)^2 \omega_r^2 + \gamma^2 \frac{\psi^2 r^2}{R^4} \right) \right)}$$

$$= \frac{\omega_z}{c^2\delta} \left\{ \frac{\partial p}{\partial z} + \left(\frac{r}{R}\right) \frac{dR}{dz} \frac{\partial p}{\partial r} \right\}. \quad (12)$$

with (9) then

$$\gamma(\ln \xi)^* = -\overset{*}{\gamma} = \frac{\omega_z}{c^2\delta} \left\{ \frac{dp_i}{dz} + \left(\frac{r}{R}\right)^2 \left(\frac{dp_E}{dz} - \frac{dp_i}{dz} \right) \right\} \quad (12')$$

Summary: under the $\mathcal{U}$-similarity & (9) & $n = n(z)$:

$$\frac{dR}{dz} = \frac{\omega_r}{\omega_z}$$

$$\omega_\varphi = \frac{\gamma\psi}{R}, \quad \overset{*}{\psi} = 0 \quad (\omega_\varphi \neq \omega_\varphi(z))$$

$\gamma_B^2 = 1 + \omega^2/c^2$

$\delta_B = \rho_i + p_E/c^2$

$\xi \neq \xi(z)$

$$\omega_r \frac{d\omega_r}{dz}\Big|_B = \frac{dR}{dz} \left\{ \frac{\gamma_B^2 \psi^2}{R^3} + \frac{2(p_i - p_E)}{\delta_B R} \right\} - \frac{\omega_r^2}{c^2 \delta_B} \frac{dp_E}{dz}$$

$$\omega_z \frac{d\omega_z}{dz}\Big|_B = \frac{1}{\delta_B} \frac{2}{R} \frac{dR}{dz} (p_E - p_i) - \frac{1}{\delta_B} \left\{ 1 + \frac{\omega_z^2}{c^2} \right\} \frac{dp_E}{dz}$$

$$R^2 \omega_z n = \text{const.}$$

Relativistic u-type jets cont:

Dick's Notation
↓
$r \equiv \frac{\tilde{\omega}}{R}$
$s \leftrightarrow l$

In the stream-line formulation

$$u_z \frac{\partial}{\partial m} \neq \frac{u}{R}\frac{\partial}{\partial r},$$

but rather

$$u_z \frac{\partial}{\partial m} = \frac{u}{R}\frac{\partial}{\partial r} - u_{\tilde{\omega}}\frac{\partial}{\partial s}$$

Check: $\frac{u}{R}\frac{\partial r}{\partial r} = \frac{u}{R}$

Now $u\frac{\partial r}{\partial m} = \frac{u_z}{R} + \frac{u_{\tilde{\omega}}}{R^2}\tilde{\omega}\frac{dR}{dz}$

$$\therefore u_z \frac{\partial r}{\partial m} = \frac{u_z^2}{uR} + \frac{\tilde{\omega}\, u_z u_{\tilde{\omega}}}{uR^2}\frac{dR}{dz}$$

and $u\frac{\partial r}{\partial s} = \frac{u_{\tilde{\omega}}}{R} - \frac{u_z\tilde{\omega}}{R^2}\frac{dR}{dz}$

$$\therefore u_{\tilde{\omega}}\frac{\partial r}{\partial s} = \frac{u_{\tilde{\omega}}^2}{uR} - \frac{\tilde{\omega}\, u_{\tilde{\omega}} u_z}{uR^2}\frac{dR}{dz}$$

$$\therefore u_z\frac{\partial r}{\partial m} + u_{\tilde{\omega}}\frac{\partial r}{\partial s} = \frac{u}{R}.$$

Now $$\omega_z^2 \frac{d}{dz}\left(\frac{\omega_{\tilde{\omega}}}{\omega_z}\right) = \frac{\gamma^2 l_s^2}{R^3} - \frac{1}{r n h c^2}\frac{u}{u_z}\frac{\partial p}{\partial m}$$

$$= \frac{\gamma^2 l_s^2}{R^3} - \underbrace{\frac{1}{r n h c^2}\left(\frac{u}{u_z}\right)^2 \frac{1}{R}\frac{\partial p}{\partial r}}_{1} + \underbrace{\frac{u\, u_{\tilde{\omega}}}{r n h c^2 u_z^2}\frac{\partial p}{\partial s}}_{2}$$

$$① = -\frac{1}{rnhc^2}\left(\frac{u}{u_z}\right)^2\left(\frac{2r}{R}\right)(p_E - p_i) = \frac{2(p_i - p_E)}{nhc^2 R}\left(1 + \left(\frac{u_{\tilde\omega}}{u_z}\right)^2\right)$$

$$② \Rightarrow u\frac{\partial p}{\partial s} = u_{\tilde\omega}\frac{\partial p}{\partial \tilde\omega} + u_z\frac{\partial p}{\partial z}$$

$$= \frac{2u_{\tilde\omega} r}{R}(p_E - p_i) + u_z\left\{\frac{dp_i}{dz} + r^2\left(\frac{dp_E}{dz} - \frac{dp_i}{dz}\right)\right.$$

$$\left. - \frac{2r^2}{R}\frac{dR}{dz}(p_E - p_i)\right\}.$$

On the boundary; $\frac{dR}{dz} = \frac{\omega_{\tilde\omega}}{\omega_z}$, $① = \frac{2(p_i - p_E)}{nhc^2 R}\left(1 + \left(\frac{dR}{dz}\right)^2\right)$,

$② = u_z \frac{dp_E}{dz}$. Hence, <u>on the boundary</u>

$$\omega_z^2 \frac{d^2R}{dz^2} = \frac{\gamma^2 l_s^2}{R^3} + \frac{2(p_i - p_E)}{(nh)_B c^2 R}\left(1 + \left(\frac{dR}{dz}\right)^2\right) + \frac{1}{(nh)_B c^2}\left(\frac{dR}{dz}\right)\frac{dp}{dz}$$

This is the same boundary eqn. as obtained by the other approach ($(nh)_B c^2 \equiv \delta_B$, $l_s \equiv \psi_s$).

As indicated prev. the extra term (pressure gradient) for an 'ideal' gas gives

$$\sim \frac{c^2}{T}\frac{dT}{dz}\frac{dR}{dz} \quad \text{for} \quad \frac{c^2}{T} \ll k/m$$

the relativistic component, but

$$\sim -\frac{3c^2}{T}\frac{dT}{dz}\frac{dR}{dz} \quad \text{for} \quad \frac{c^2}{T} \gg k/m$$

the 'thermal' component.

k/m
$\sim 10^{11}$ elect.
$\sim 10^{8}$ prot.
cm^2/sec^2
$\frac{mc^2}{k} \sim 10^{10}$ el.
$\sim 10^{13}$ p.