RELATIVISTIC ELECTRODYNAMICS

A. H. BRIDLE

Trinity College
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Summary The matrix formulation of special relativity will be applied to the foundations of electrodynamics; the approach follows the principle of the previous article on mechanics in seeking to express all physical laws in terms of 4-vectors. The extension of the principle to 4-tensors will be used, retaining the matrix formulation.

Introduction

In the article "Transformation of Co-ordinates in Special Relativity", to be referred to henceforth as (I), it was shown that the postulates of special relativity lead to a transformation between inertial observers embodied in the 4x4 orthogonal matrix (L) with $L_{11} = \beta$, $L_{22} = \beta$, $L_{12} = -iv\beta/c$, $L_{21} = iv\beta/c$, $L_{33} = L_{44} = 1$, and all other terms zero.

In this representation, events in the space-time manifold are expressed as vectors (ict,x,y,z), and all inertial frames are taken in standard configuration.

In the article "The Formulation of Mechanics in Special Relativity", to be referred to henceforth as (II), the principle that all physical laws should be expressed in terms of vectors transforming like the event-vectors was used to derive a system of mechanics, the validity of which was established by appeal to experiment. Electrodynamics will now be approached in the same spirit; it is to be hoped of course that the verification of the method achieved by reference to practice in the case of mechanics will anticipate a similar result here, leading to a unification in relativistic physics.

4-vector Operators

In all of electrodynamics we are concerned with vector fields, i.e. quantities of a vectorial nature which vary from point to point in space, and the calculus of vectors is needed for the expression of physical laws. The discussion of the relativistic form must begin then with the transformation properties of the calculus operators.

Consider the column

$$\Box = \begin{pmatrix} \partial/\partial(ict) \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

, a vector operator in system S.

In another inertial system S' its components will take the form

$$\frac{\partial}{\partial(ict')} = \frac{\partial}{\partial(ict)}.\frac{\partial t}{\partial t'} + \frac{\partial}{\partial x}.\partial x/\partial t' + \frac{\partial}{\partial y}.\partial y/\partial t' + \frac{\partial}{\partial z}.\partial z/\partial t'$$

But $dt = L_{11}dt' + L_{21}dx' + L_{31}dy' + L_{41}dz'$

$dx = L_{12}dt' + L_{22}dx' + L_{32}dy' + L_{42}dz'$

... etc , from $X = (\bar{L})X'$, as $(\bar{L}) = (L)^{-1}$

Hence $\frac{\partial}{\partial(ict')} = L_{11}\frac{\partial}{\partial(ict)} + L_{12}\frac{\partial}{\partial x} + L_{13}\frac{\partial}{\partial y} + L_{14}\frac{\partial}{\partial z}$, i.e. $\Box' = (L)\Box$, the other components transforming similarly.

From this it follows that the operator $\Box^2 = \nabla^2 - (1/c^2).\frac{\partial^2}{\partial t^2} = \bar{\Box}\Box$

is Lorentz invariant, i.e. $\bar{\Box}'\Box = \bar{\Box}(\bar{L})(L)\Box = \bar{\Box}\Box$

Thus the column represented by $\Box$ transforms like a 4-vector, and may be termed a 4-operator by analogy, and $\Box^2$ transforms like a scalar.

4-Tensors

It will be necessary to extend the idea of a 4-vector to include entities corresponding to $\underline{\nabla}\times\underline{V}$ (= curl $\underline{V}$) in three-dimensional vector calculus.

In general tensor theory, a quantity T_{kl} is tensorial if the representation T'_{ij} under a transformation of co-ordinates with matrix M_{ab} is obtained through

$$T'_{ij} = M_{ik}M_{jl}T_{kl} = M_{ik}T_{kl}\bar{M}_{jl} \qquad \text{(sc)}$$

For a second-rank tensor, with two suffices, this transformation may be written

$$T' = (M)T(\bar{M})$$

We shall say that a quantity is a 4-tensor if its transformation law is

$$T' = (L)T(\bar{L})$$

It follows by an argument similar to that employed in (II), that 4-tensors are suitable quantities for the expression of physical laws in special relativity.

To construct a quantity similar to curl $\underline{V}$, consider the set of quantities G_{ij} formed from a 4-vector V by the following rule:

$$G_{ij} = (\partial V_i/\partial x_j) - (\partial V_j/\partial x_i)$$

$$\text{i.e.} \quad (G) = \Box\bar{V} - \overline{(\Box\bar{V})}$$

In a system S' we have that

$$\begin{aligned}(G)' &= \Box'\bar{V}' - \overline{(\Box'\bar{V}')}\\ &= (L)\Box\bar{V}(\bar{L}) - (L)V\bar{\Box}(\bar{L})\\ &= (L)(\Box\bar{V} - \overline{(\Box\bar{V})})(\bar{L})\\ &= (L)(G)(\bar{L})\end{aligned}$$

Thus the quantities G_{ij} make up a second-rank 4-tensor expressible as a (skew-symmetric) 4 x 4 matrix (G).
These quantities will be defined as the "4-curl" of the vector V.

The tensor G has an interesting identical relation holding between its components, which arises from its manner of formation. This is that

$$\partial G_{ij}/\partial x_k + \partial G_{jk}/\partial x_i + \partial G_{ki}/\partial x_j \quad \text{is identically} = 0$$

This can readily be verified by performing the differentiations and summing. Every term of the form $\partial^2 V_i/\partial x_j\partial x_k$ occurs twice, once with a + and once with a - sign, so that the sum vanishes identically. This property, which is purely one of the structure of the particular type of 4-tensor typefied by G, will be of importance later in the theory.

4-current Density

The expression of the notion of charge conservation in three-dimensional electrodynamics is the "equation of continuity"

$$\underline{\nabla}\cdot\underline{j} + \dot{\rho} = 0$$

If we define a vector

$$\mathcal{J} = \begin{pmatrix} ic\rho \\ j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} ic\rho \\ \underline{j} \end{pmatrix} \text{ in partitioned form,}$$

the equation of continuity becomes simply $\bar{\Box}\mathcal{J} = 0$. Now the idea of conservation of charge is evidently not one which can be correlated with the properties of any inertial frame if all such frames are to be equivalent for the expression of physical laws, so we must find in a frame S' the equation

$$\bar{\Box}'\mathcal{J}' = 0$$

where $\mathcal{J}'$ is the corresponding vector in S'. But we know that $\Box' = (L)\Box$, so that the equation may be written as $\bar{\Box}(\bar{L})\mathcal{J}' = 0$. Now it is a sufficient, but not a necessary, condition for this to be satisfied that

$$\mathcal{J}' = (L)\mathcal{J} \quad \text{, i.e. that } \mathcal{J} \text{ is a 4-vector.}$$

This clearly recommends itself as a hypothesis in the spirit of the general principle of 4-vector formulation of the theory, and we shall therefore take the vector $\mathcal{J}$ to transform via the Lorentz matrix. It will be called the "4-current density" and combines the non-relativistic current density and charge density in one vector; this unification of the representation of apparently distinct non-relativistic quantities is one of the prominent characteristics of the theory. It stems from the initial step of representing events by a vector $X = (ict, x, y, z)$ (as in (I)); an example may be drawn from (II), where we found the relativistic analogue of momentum to be $P = M.dX/dT$, where M is the rest-mass of a particle, and dT the element of proper time. The "space-like" components of this vector give the quantity $\beta M\underline{v}$, while the "time-like" component is $i\beta Mc$, or iE/c where $E = Mc^2 \times \beta$. Thus the 4-momentum combines the concepts of energy and momentum from non-relativistic physics, although in the modified form including the factor β.

<u>The 4-Potential</u>

The field potentials of non-relativistic electrodynamics are related to the current and charge distributions in space through the relations :

$$-\frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} + \nabla^2 A_i = -j_i/c \qquad -\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2\phi = -\rho$$

Here the A_i are the components of a three-dimensional vector potential $\underline{A}$. We see that if we define a four-dimensional vector $\mathcal{A} = (i\phi, \underline{A})$, these equations may be combined in the single equation

$$\Box^2\mathcal{A} = -\mathcal{J}/c \quad \text{, using the 4-current density } \mathcal{J} \text{ from above.}$$

Now if $\mathcal{A}'$ and $\mathcal{J}'$ are the vectors corresponding to $\mathcal{A}$ and $\mathcal{J}$ in a frame S', we must have

$$\Box'^2\mathcal{A}' = -\mathcal{J}'/c \quad \text{in that frame}$$

$\Box'^2 = \Box^2$, $\mathcal{J}' = (L)\mathcal{J}$, $\Box^2\mathcal{A}' = -(L)\mathcal{J}/c$, which is the same as $\Box^2\mathcal{A} = -\mathcal{J}/c$ if we have

$$\mathcal{A}' = (L)\mathcal{A}$$

This is again a sufficient but not a necessary condition for the Lorentz invariance of the equation concerned, but as it involves the hypothesis that $\mathcal{A}$ is a 4-vector it will be adopted. We thereby continue to unify the representation by expressing both the vector potential $\underline{A}$ and the scalar potential ϕ of non-relativistic theory in one 4-vector.

<u>The Field Tensor and Maxwell's Equations</u>

The vector and scalar potentials generate the field quantities $\underline{E}$ and $\underline{H}$ through the equations :

$$\underline{E} = -\underline{\nabla\phi} - \frac{1}{c}.(\partial\underline{A}/\partial t) \qquad \underline{H} = \underline{\nabla x A} \qquad \text{(in free space)}$$

Inspection shows that these formulae are equivalent to the rule given earlier for the construction of a skew-symmetric 4-tensor G_{ij} from a four-vector V, where now $V = \mathcal{A}$.

Thus if we form the tensor $F_{ij} = \Box\bar{\mathcal{A}} - \overline{(\Box\bar{\mathcal{A}})} = \Box \times \mathcal{A}$ schematically

$$F = \begin{pmatrix} 0 & iE_x & iE_y & iE_z \\ -iE_x & 0 & H_z & -H_y \\ -iE_y & -H_z & 0 & H_x \\ -iE_z & H_y & -H_x & 0 \end{pmatrix}$$

Thus the tendency towards unification found in the theory so far is continued when we construct the relativistic form of the field vectors, except that we now find the field quantities intermingled in a single skew-symmetric 4-tensor.

We have now relativistic analogues or representations of all the elements of non-relativistic electrodynamic theory, the charge and current densities, the scalar and vector potentials, and now the field vectors themselves. In each case the familiar non-relativistic quantities have been seen to play dualistic roles with one another in some way. The "mixing" of the field quantities $\underline{E}$ and $\underline{H}$ in this theory is well illustrated by the transformation equations for field quantities between frames of reference ; these are derived simply from the law of 4-tensor transformation.

If in a frame S we measure the field quantities $\underline{E}$ and $\underline{H}$, we may represent these in a tensor F. In a frame S', the corresponding $\underline{E}'$ and $\underline{H}'$ are found in a tensor F' where

$$F' = (L)F(\overline{L})$$

The matrix (L) for two frames in standard configuration with relative motion along the common x-axes was found in (I), whence

$$F' = \begin{pmatrix} \beta & -iv\beta/c & 0 & 0 \\ iv\beta/c & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & iE_x & iE_y & iE_z \\ -iE_x & 0 & H_z & -H_y \\ -iE_y & -H_z & 0 & H_x \\ -iE_z & H_y & -H_x & 0 \end{pmatrix} \begin{pmatrix} \beta & iv\beta/c & 0 & 0 \\ -iv\beta/c & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Giving the equations

$$E'_x = E_x \qquad H'_x = H_x$$
$$E'_y = \beta(E_y - vH_z/c) \qquad H'_y = \beta(H_y + vE_z/c)$$
$$E'_z = \beta(E_z + vH_y/c) \qquad H'_z = \beta(H_x - vE_y/c)$$

The mixing of electric and magnetic phenomena in these formulae provides a satisfying view of the observed inter-relation between electric and magnetic interactions in the world at large. In particular, the effect of a magnetic field on a moving electric charge achieves some theoretical basis ; for if a charge moves along the x axis initially, and a magnetic field H_z is applied in the rest-frame of an observer, the transformation equations tell us that in the rest-frame of the charge there is an electric field E'_y, of magnitude $v\beta H_z/c$. Thus the charge will be deflected in the y-direction by this electric field. The deflection in the y-direction as seen by the observer at rest however, cannot be attributed to any electric field - it is ascribed to the hitherto mysterious "Lorentz force" of the form $\underline{v}\text{x}\underline{H}/c$. The theory we are developing here then clearly has the merit of predicting the appearance of this force when charges move in the presence of magnetic fields - it is the electric field in the rest-frame of the charge which causes it to be deflected.

It was remarked earlier that four identical relations hold between the components of a skew-symmetric four-tensor, viz.

$$\partial G_{ij}/\partial x_k + \partial G_{jk}/\partial x_i + \partial G_{ki}/\partial x_j \text{ identically} = 0, \text{ where } i,j,k = 1,2,3;\ 2,3,4;\ 3,4,1;\ 1,2,4$$

With the tensor F_{ij} above, and $x_1 = ict$, $x_2 = x$, $x_3 = y$, $x_4 = z$, these four identities become

$$\underline{\nabla}.\underline{H} = 0 \quad \text{and} \quad \underline{\nabla}\text{x}\underline{E} = -\frac{1}{c}.\partial\underline{B}/\partial t$$

These are four of the non-relativistic Maxwell Equations for the electromagnetic field, though. As such tensor identities obviously hold in any frame of reference, being a property only of the structure of the tensor concerned, it follows that these two of Maxwell's Equations are Lorentz invariant ; i.e. if they hold unprimed in S, they hold primed in S'.

It may be verified directly that the other pair of Maxwell Equations in ordinary electrodynamics may be expressed in the form

$$\bar{\Box}\bar{F} = +\bar{\mathfrak{J}}/c$$

The very possibility of representing them in this form is sufficient to ensure their Lorentz invariance in form, as the only quantities appearing in them are 4-vectors, a 4-tensor, and the scalar invariant c. Thus we find that Maxwell's Equations are Lorentz invariant.

Comment on the Invariance of Maxwell's Equations.

This result of invariance of the equations of electrodynamics under a Lorentz transformation is scarcely surprising - it is a well-known property of the equations that they predict a definite velocity for electromagnetic radiation in vacuo, independent of the system of measurement. This very result is in conflict with pre-Lorentz relativity, but is of course the mainspring of the present formalism. The invariance we have discovered merely shows us that the formalism is consistent within itself. As a result, electrodynamics in its classical form comes through the new discipline unscathed, unlike classical mechanics, where we found that the laws of motion took on a new detailed form. The value of relativistic electrodynamics lies therefore not in its better correspondence with experimental fact by production of more exact equations but in the unification of concepts that are somewhat diverse in the classical theory. The transition to 4-dimensional formalism brings with it a more compact and illuminating descriptive system, which is of value in itself.

Field Invariants

The properties of the field tensor F_{ij} will next be used to examine some of the transformation properties of the fields $\underline{E}$ and $\underline{H}$.

Two invariant quantities can be formed by contraction of the tensor F_{ij}, namely

$$F_{ij}F_{ij} \qquad \text{and} \qquad \varepsilon_{ijkl}F_{ij}F_{kl}$$

(The invariant F_{ii} is clearly = 0)

Taking these quantities in their three-dimensional form, we find that

$$E^2 - H^2 = \text{scalar invariant} \quad \text{and} \quad \underline{E}.\underline{H} = \text{scalar invariant}.$$

From these results the following properties of the field transformations may be deduced immediately :

i) if in any frame S, the fields $\underline{E}$ and $\underline{H}$ are perpendicular, then $\underline{E}.\underline{H} = 0$. The invariance of this quantity therefore implies that $\underline{E}'$ and $\underline{H}'$ will be perpendicular in all other inertial frames S'.

ii) if in any S $\underline{E}.\underline{H} = 0$, it must be possible to find some inertial frame in which E or H = 0, i.e. if in some S $\underline{E}$ and $\underline{H}$ are perpendicular, it is possible to find an inertial frame in which one or the other vector vanishes. Conversely, if in some S one or the other vector vanishes, $\underline{E}'$ and $\underline{H}'$ will be perpendicular in all other frames S'.

iii) relative relations between E and H are preserved in all frames. For instance, if in some S E is greater than H, it must be so in all inertial frames, from th first invariant. Similarly, vice versa, and similarly equality between the magnitudes of the vectors is common to all frames.

The Lorentz Force

We have already seen that in the limit of small relative velocities, the transformation laws for the electric and magnetic fields give rise to a term equivalent to the "Lorentz force" of classical electromagnetism. We shall now consider this in the language of the 4-dimensional representation.

The Lorentz force is a quantity describing the interaction between currents and fields. We should obviously wish to describe it in terms of a 4-force vector $\mathfrak{F}$. We can see that this is possible from the field transformation equations and the definition of a 4-force arrived at in (II). There we saw that a 4-force $\mathfrak{F}$ is related to the 3-force f as $\mathfrak{F} = \beta f$, which is exactly the form implied by the transformation equations for the fields, in terms of the transformation of a magnetic field in a frame in which a charge is moving into an electric field in a frame in which it is stationary. To make this precise, let us consider the most likely form for the 4-force arising from the interaction of fields and currents. Clearly it will be a vector derived from the matrix F and the 4-vector $\mathfrak{J}$. The simplest way of forming a 4-vector $\mathfrak{F}$ from the quantities F and $\mathfrak{J}$ is clearly to put

$$\mathfrak{F} = F\mathfrak{J}$$

That $\mathfrak{F}$ defined in this way is a 4-vector is easily shown by the methods employed earlier in this article. Now the matrix F is, in partitioned form,

$$F = \begin{pmatrix} 0 & i\overline{E} \\ -iE & H^* \end{pmatrix}$$

where H* denotes the 3 x 3 skew-symmetric matrix associated with the 3-vector H through the rule $H^*_{ij} = \varepsilon_{ijk}H_k$. Then $\underline{H} \times \underline{V}$ in 3 dimensions is simply H*V in matrix form.

Thus the vector $\mathfrak{F}$ is given by

$$\mathfrak{F} = \begin{pmatrix} 0 & i\overline{E} \\ -iE & H^* \end{pmatrix} \begin{pmatrix} ic \\ \underline{j} \end{pmatrix}$$

$$= \begin{pmatrix} i\overline{E}j \\ cE - H^*j \end{pmatrix}$$

Now the Lorentz force per unit volume in the limit of small velocities is $\underline{E} - \frac{1}{c}\underline{H} \times \underline{j}$, so we see that we can construct the Lorentz 4-force through the rule

$$\mathfrak{F} = F\mathfrak{J}/c$$

The "time component" of this 4-force is then the rate of working of the electric field.

The Liénard-Wiechert Potentials

The appearance of the familiar Lorentz force in slightly modified form (the modification corresponding to the change in the definition of force in mechanics, rather than anything intrinsic to electrodynamics) is just one example of the general preservation of classicial theory in the relativistic form. Once Maxwell's equations are seen to be invariant, no more need really be said, except to point out the beauty of the four-dimensional structure. (The formal basis of electrodynamics includes of course the arbitrary gauge restriction - the conventional gauge $\underline{\nabla}.\underline{A} + (1/c)\partial\phi/\partial t$ is clearly written $\Box\mathcal{A} = 0$ in 4-form, which is Lorentz invariant by inspection). Results such as the form of the field-current interaction, and the reppresentation of energy flow in the Poynting vector, will clearly be carried over into the relativistic theory. One last point will be considered, however, and that is the form of the potential produced by a moving charge. Suppose we observe a charge moving in a frame S. Let S' be the frame in which the charge is at rest. Then, if q' is the magnitude of the charge, our assumptions of the 4-vector nature of $\mathfrak{J}$, together with the equation of continuity, imply that q' = q.

The vector $\mathcal{A}'$ is seen to be (iq/r', 0, 0, 0)

It follows from the 4-vector nature of $\mathcal{A}$ that the corresponding potential 4-vector in S is given by

$$\mathcal{A} = (i\beta q/r', \beta vq/cr', 0, 0)$$

This still contains the quantity r', however, and to obtain the potential seen in S, we must transform this to the quantity r, and possibly other parameters measured in S.

To do this, we note that the potentials at a point in S will depend at any instant not on the position of the charge at that instant, but at an earlier instant given by $r = -ct$. In S', similarly we require $r' = -ct'$. This enables us to transform r' out of the equation for the potentials by using the transformation for t', thus :

$$r' = -ct' = -c\beta(t - vx/c2) = r\beta(1 + v_r/c)$$, where v_r is the radial velocity of q as seen in S, $= vx/r$.

Thus we can derive the potentials in the form

$$A_x = (q/c)\cdot\left[\frac{v}{r(1 + v_r/c)}\right] \qquad A_y = A_z = 0$$

$$\phi = \frac{q}{[r(1 + v_r/c)]}$$, where square brackets signify retarded values.

Two generalisations of these formulae can be made. The first uses the fact that we know the choice of "standard configuration" for two inertial frames not to sacrifice generality, as was shown in (I). Therefore we may generalise the result obtained for the particular case of uniform velocity along the x-axis to the equations:

$$\underline{A} = \frac{1}{c}\left[\frac{q\underline{v}}{r + \underline{v}.\underline{r}/c}\right] \quad \text{and} \quad \phi = q.\frac{1}{[r + \underline{v}.\underline{r}/c]}$$

The second generalisation is perhaps more fundamental. We know from the ordinary three-dimensaional electrodynamics that the potentials due to a moving charge depend only on its velocity, and not on its acceleration. This fact enables us to generalise the above proof to cover the case of an arbitrarily moving charge ; our proof took a uniform velocity $\underline{v}$ (for which the retarded value is of course the same at all times). If the potentials do not depend on the acceleration, then the equations found above give us the correct potentials at any instant in S, so long as we use the correct retarded value of the velocity $\underline{v}$ of the charge - in other words, the calculation is correct for the instantaneous inertial frame in which the charge may be considered at the appropriate retarded value of t. The knowledge of the dependence of the potential on the acceleration of a charge would of course invalidate this generalisation.

The potentials here derived for an arbitrarily moving charge are known as the Lienard-Wiechert potentials.

Conclusion

The expression of electrodynamics obtained by application of the methods of articles (I) and (II) has achieved results different in kind from those found in (II) for relativistic mechanics. The basic equations are found to be unmodified, as is consistent with the constant velocity of propagation of electromagnetic radiation implied by Maxwell's equations in their usual form. The bringing together of similar but differentiated quantities in the three-dimensional theory is achieved by pursuing the 4-dimensional approach, and a degree of compactness in writing the equations can thus be achieved also. The derivation of the Lienard-Wiechert potentials shows that, in problems where the measurement of distance and time appears in the solution, the final results will be modified; this modification occurs entirely through the transformation of co-ordinates, however, and is not one arising from the transformation properties of the electromagnetic quantities themselves.

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