APPLES, TIDES, COMETS AND THE MOON
--UNIVERSAL GRAVITATION

The prohibition of Copernican teachings by the Roman Church forced the development of physical science into North-western Europe, where the next great steps were taken by the French, Dutch and English. The decades from 1640 onwards to the end of the Seventeenth Century are regarded as the dawn of the modern scientific method, leading to the brilliant daylight of Sir Isaac Newton's "Mathematical Principles of Natural Philosophy", which can still be described as the most influential scientific treatise ever written. This period, known to historians as the Age of Enlightenment, saw the application of the experimental method and mathematical logic to many scientific fields and witnessed the development of the philosophy of the Universe as a mechanical system, describable by the same laws of physics that were embodied in the design of terrestrial machines. We shall be concerned only with those elements of scientific progress in this period which relate directly to the problems of the Solar System and of cosmology, but it is well to realise that the scientists who grappled with these problems were also attempting to apply their new-found skills in other areas of research, such as optics and the behaviour of gases and fluids.

1. Rene Descartes (1596-1650)

The Frenchman Rene Descartes (Figure 1) was the first to construct a comprehensive philosophy rejecting every precept of the Aristotelians. He argued that motion was not directed towards some goal by animalistic desires of matter, but was an entirely mechanical, or mindless attribute of matter that could be completely described by changes in its co-ordinates in space with time (Descartes was the inventor of co-ordinate geometry in the form that we use it throughout this book). He argued also that the total "quantity of motion" in the material world was preserved, as it would be in a frictionless machine.

He and his followers saw the Universe as a complex machine whose workings followed mathematical principles, and whose phenomena should all be explicable as some form as motion--He wrote:

> "We must conclude on all counts that the objective external realities that we designate by the words 'light', 'colour', 'odour', 'flavour', or 'sound', or by the name of tactile qualities such as 'heat' and 'cold' ... are not recognisably anything other than the powers that objects have to set our nerves in motion in various ways".

Descartes viewed the human body itself as a machine, e.g. he construed the circulation of the blood through the heart to be the mechanical action of a machine devoid of mystical qualities and describable (if not actually described) by mathematical models. Because he considered all that was dynamic in the world to be a consequence of the motion of matter, it was crucial to his world-view that matter as a whole preserved its "quantity of motion".

Descartes argued that motion could be transferred from one body to another, but that the total quantity of motion imbued in the Universe at its creation was conserved. The agency of change in motion was force, and the measure of force was the change in motion which it produced. Within this framework, Galileo's deduction that a projectile in motion preserved its horizontal velocity finally received its valid generalisation--that matter has an attribute of "inertia"--reluctance to change its motion rather than desire to seek a certain place. Anyone who has tried to bring a moving boat into dock, to maneouvre a heavy packing case on a trolley running on smoothly-rolling casters, or to drive a car around a tight corner on an icy road, will recognise the attribute of matter which Descartes regarded as fundamental. A force--a push or a pull--is needed to change motion, not to maintain it, in the absence of friction. The amount of force needed to change motion depends both on the quantity of matter moving and on the speed (i.e. velocity) with which it moves. The idea of quantity of matter is related to the word "mass", and so we shall use the symbol "m" to denote the measure of quantity of matter; the precise definition of this measure must wait a while. If "quantity of motion" increased both with increasing mass and increasing velocity, then what algebraic form should measure "quantity of motion"? We shall see that both momentum (mv) and energy ($\frac{1}{2}mv^2$) provide meaningful measures. Descartes believed that our qualitative concept of force should be related to changes in the quantity of motion as measured by momentum (mv).

He also recognised that changes in direction of travel were as much changes in motion as were changes in speed. To clarify this point, we can reconsider Galileo's projectile experiment, in which a ball was rolled down a ramp and launched horizontally across the room. As the ball leaves the deflector travelling horizontally, it has no vertical motion. But whilst it continues to travel at constant velocity horizontally, it experiences a vertical acceleration--its vertical motion changes. Descartes' description of the forces in this situation would therefore be exactly opposite to the Aristotelian version, in which there was no force associated with the natural (vertical) motion but a steady force (of unknown origin) was supposed to propel the horizontal motion. Descartes would have argued that there was a force (of unknown origin) accelerating the <u>vertical</u> motion, while the horizontal motion was forceless and maintained by matter's property of "inertia". The vertical force changes the vertical motion by changing (increasing) the vertical velocity. The projectile therefore moves on a steepening (parabolic) curve: its total (two-dimensional) velocity is the combination of its horizontal and vertical <u>components</u>, as in Figure 2a. The total velocity therefore changes both magnitude <u>and</u> direction as the projectile falls. In modern mathematical language we say that a quantity which carries both magnitude and direction information is a <u>vector</u>. The total velocity of Galileo's projectile is describable by a two-dimensional velocity vector which we will call $\vec{v}$, the arrow over the symbol reminding us that direction information is being carried in the symbolism. The $\vec{v}$ of the projectile at any instant can be regarded as the <u>sum</u> of a horizontal velocity component v_x, which supplies information about the horizontal displacement Δx which will occur in the next Δt, and a vertical velocity component v_y, which supplies information about the vertical displacement Δy which will occur in the next Δt. If we regard the rule for two-dimensional <u>vector addition</u> as being that embodied in the "vector diagram" of Fig. 2b then we can write

$$\vec{v} = \frac{\Delta \vec{s}}{\Delta t}$$

where $\Delta \vec{s}$ is the <u>vector</u> displacement (change in position) in time Δt, and for the projectile

$$v_x = \frac{\Delta x}{\Delta t} = \text{constant (Galileo)}; \quad a_x = \frac{\Delta v_x}{\Delta t} = 0$$

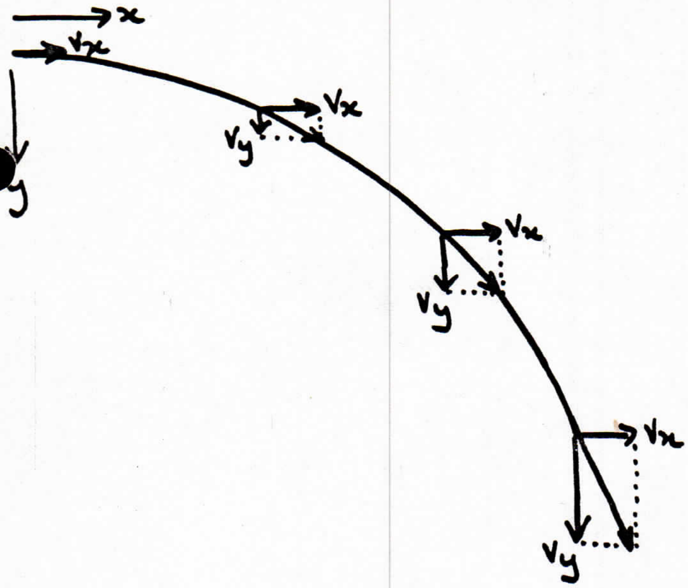
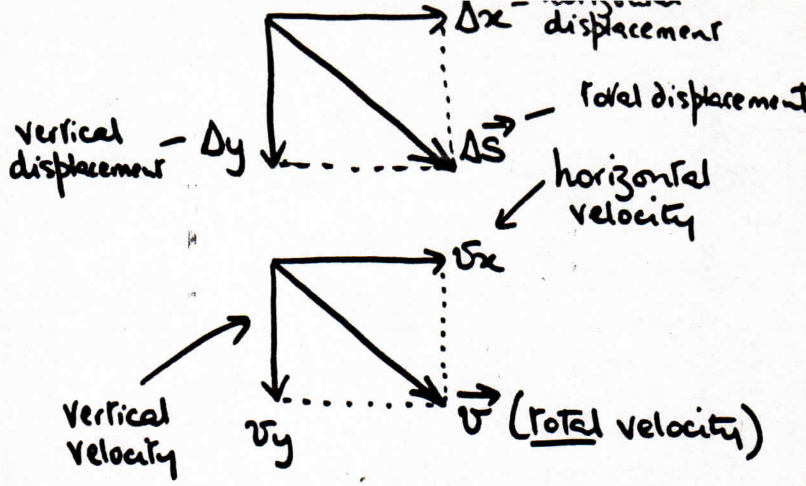


Figure 2a



2b

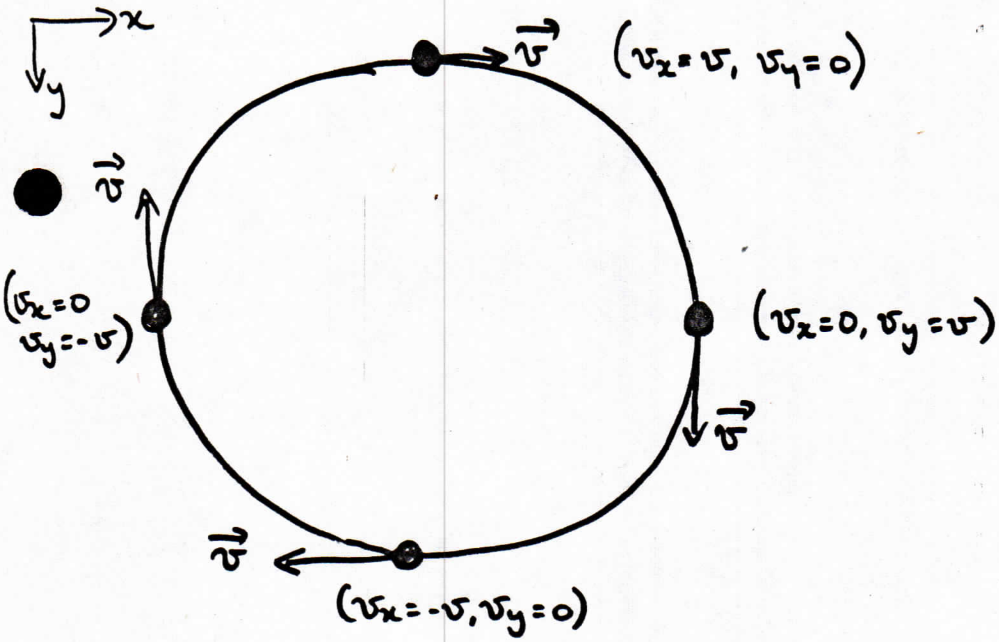


Fig 3

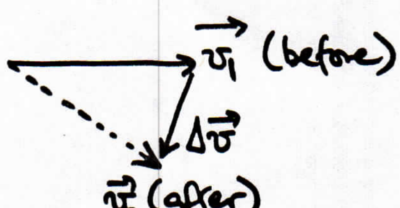


Fig. 4

Fig 5

Indeed, anything other than uniform motion along a straight line (or rest) is a motion in which $\vec{v}$ changes in time, and so implies accelerations and forces.

Descartes was only a step away from Newton's thinking in these matters. It is unfortunate that Kepler's Laws, particularly the Third Law, had no currency in France for most of Descartes' career, so yet another brilliant mathematical mind was kept from analysing the Solar System in the way warranted by Tycho Brahe's data.

2. Christian Huygens (1629-1695)

The most prominent Dutch disciple of Descartes was Huygens, who is most renowned today for his studies in the nature of light and in the design of clocks. Huygens did however make a most important contribution to the mathematical description of motion--namely the formula for the acceleration of a body moving uniformly on a circle. Newton obtained this result independently but it was Huygens who first published it, in 1673.

The most difficult aspect of the problem is to see the direction of the acceleration. Acceleration is also a vector quantity; it is the rate of change of velocity, which is a vector quantity. If a vector velocity $\vec{v}$ changes by an amount $\Delta\vec{v}$ (see Figure 4), then

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

Referring back to Figure 3 we can see that both v_x and v_y change during a uniform circular motion, so that $\vec{a}$ must have both x and y components. To disentangle the problem, we need to consider a very short time-interval Δt during the motion. In that time, $\vec{v}$ changes from $\vec{v}_1$ to $\vec{v}_2$ as in Figure 5; $\vec{v}_2$ has the same magnitude (speed) as $\vec{v}_1$, but is in a slightly different direction. By the rule for vector addition, $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$ where $\Delta\vec{v}$ is the change in velocity indicated in Figure 5. As $\Delta t \to 0$, $\Delta\vec{v}$ becomes more and more precisely at right angles to $\vec{v}_1$ and towards the centre of the circle. Huygens indeed deduced that a body travelling around a circle at uniform speed has a constant acceleration in the direction towards the centre of the circle. This centripetal (centre-seeking) acceleration changes direction continuously as the body travels around the circle, but is always at right angles to the velocity $\vec{v}$. Thus, in Figure 3, when v_x is a maximum and $v_y = 0$ it is v_y which is changing; at this point $a_x = 0$ and a_y is a maximum.

Huygens also calculated how the centripetal acceleration varies with the magnitude of the velocity v and the radius r of the circle; the derivation is given in Appendix 1. The result is that

$$a = \frac{v^2}{r}$$

It will be useful to discuss this result using a modern example--the motor car. Now that we have arrived at careful definitions of velocity and acceleration you may realise that every car has <u>three</u> accelerators as standard equipment, i.e. three controls which you can use to change its velocity. There is the one we conventionally call "the" accelerator--the gas pedal. Providing more fuel to the engine increases the force which the engine can transmit to the drive wheels, thus making the car go faster in a straight line. The $\Delta\vec{v}$ produced is <u>parallel</u> to the $\vec{v}$, so the speed <u>increases</u> with time. But the brake pedal is also an accelerator. By increasing friction at the wheels when the brake linings are pressed against a moving surface we introduce forces which produce a $\Delta\vec{v}$ that <u>opposes</u> $\vec{v}$, so the speed <u>decreases</u> with time. This is sometimes called "deceleration": using the scientific language of motion it is just an acceleration in the opposite direction from the one produced by pressing down on the gas pedal.

The third accelerator--and this is the one related to Huygens' idea--is the steering wheel. If you wish to drive around a bend in the road at constant speed you must change the <u>direction</u> of the car's velocity, something neither the gas pedal nor the brake can accomplish. When you turn the steering wheel you apply a force which turns the front wheels of the car and if there is sufficient friction between these wheels and the road this results in a sideways force on the car. If your tires are too bald, or the road is too slippery, you discover the principle of inertia as you travel straight ahead at constant speed and fail to turn the <u>corner</u> despite the fact that you <u>have</u> turned the front wheels. If you have good tires and the road is in good condition, the force between the tires and the road can give your car an acceleration towards the <u>inside</u> of the bend, changing your car's velocity so that it can move in the same direction as the road. If you look at Huygens' formula for the amount of the acceleration you will see that it embodies some well-known experiences in driving. If your tires have a given amount of friction with the road you can provide a given acceleration "a" with a turn of the steering wheel. If you go into a bend of radius r travelling at a certain v, this acceleration may not be sufficient to turn the bend --the acceleration <u>needed</u> goes up as v^2. If you exceed the maximum v for a given r by even a small amount, you will fail to change your car's direction of travel enough to curve around with the road. If the "safe speed" is v = 20 miles per hour and you try to travel around the curve 20 miles per hour too fast, i.e. at 40 miles per hour, then the acceleration you <u>need</u> is $(40)^2 \div (20)^2$, or four times, greater than what you can safely provide, even with much squealing of tires. But if the safe speed is v = 60 miles per hour and you again exceed it by 20 miles per hour, the acceleration is only $(80)^2 \div (60)^2$, or about 1.8 times greater than you can safely provide, and you may "make it". Finally, the "safe speed" depends on r, the radius of the bend. If r is small, than $a = v^2/r$ is large even for small v--tight corners have to be taken slowly. But if r is large, $a = v^2/r$ is small except at very high speeds--gentle curves are easy to negotiate. You can see that driving cars gives an intuitive "feel" for Huygens' centripetal-acceleration formula.

With this formula, those who were lucky enough to <u>find</u> Kepler's Third Law in the "Harmony of the Worlds" were in a position to understand what everybody today knows as "the force of gravity".

3. The Inverse Square Law of Gravity

The Royal Society for the Promotion of Natural Knowledge was founded in London, England in 1661, its nucleus being a group of scholars who had been meeting at Wadham College, Oxford throughout the previous decade to discuss controversial matters of science. The first Curator of Experiments was Robert Hooke (1635-1703), a follower of Descartes' mechanistic views who made important contributions to the sciences of elasticity and optics in addition to his work on gravity (as the vertical acceleration of bodies near the Earth's surface was now called). A matter of some concern to the members of the Royal Society was whether or not the acceleration due to gravity varied with height above or below the Earth's surface--Galileo's result that it did not depend on the <u>mass</u> of the accelerated body was now well known. On December 3, 1662 one Dr. Power reported to the Society the results of an experiment wherein a pound weight and 68 yards of thread had been weighed on a balance. The weight was then lowered on the thread into an open mineshaft while the balance remained at ground level (Figure 6); Dr. Power claimed that the balance recorded a loss in weight of about an ounce, suggesting that gravity decreased rapidly towards the centre of the Earth. Three weeks later Robert Hooke attempted a similar experiment and reported it to the Royal Society thus:

> "I took an exact pair of scales and weights, and went to a convenient place upon Westminster Abbey ... Here counterpoising a piece of iron (which weighed about 15 ounces troy) and packthread enough to reach from the top to the bottom, I found the counterpoise to be of troy-weight seventeen ounces and thirty grains. Then letting down the iron by the thread ... I tried what alteration there had happened to the weight, and found that the iron preponderated the former counterpoise somewhat more than ten grains. Then drawing up the iron and thread with all the diligence possibly I could, that it might neither get nor lose anything by touching the perpendicular wall, I found by putting the iron and packthread again into its scale that it kept its last equilibrium; and therefore concluded that it had not received any sensible difference of weight from its nearness to or distance from the Earth. I repeated the trial in the same place but found that it had not altered its equilibrium (as in the first trial) neither at the bottom, nor after I had drawn it up again; which made me guess that the first preponderating of the scale was from the moisture of the air, or the like, that had stuck to the string and so made it heavier."

Hooke went on to suggest that Power's experiment might have been influenced by air moisture or drafts. The Royal Society members continued experiments in this vein for several years. The experiments show two things: that the idea that the gravitational acceleration might vary with height was quite public, and that the actual variation near Earth's surface was known to be very small.

The members of the Society were also appraised of Kepler's Third Law of Planetary Motion and at some time between Huygens' publication of the formula for centripetal acceleration and the year 1679 it appears that the following calculation became known, at least to Robert Hooke, his close friend the architect Sir Christopher Wren, and the astronomer Edmond Halley.

Kepler's Third Law refers to the time taken to perform <u>elliptical</u> orbits of various sizes around the Sun. The actual ellipses are, however, very close to circles for most of the planets, so Huygens' formula for the centripetal acceleration on a circle should <u>approximately</u> give the planetary accelerations. Thus a planet travelling with velocity v on an orbit which was nearly a circle of radius r would have an acceleration

$$a = \frac{v^2}{r} \quad \underline{\text{towards the Sun}}$$

But an orbit of radius r would have a total length that was equal to the circumference of the circle, i.e. $2\pi r$, and be performed in time T. The magnitude of the velocity of each planet around the Sun would thus be

$$v = \frac{\text{Circumference of Orbit}}{\text{Orbital period}} = \frac{2\pi r}{T}$$

The acceleration of each planet could therefore be written in terms of the observable properties r and T:

$$a = \frac{v^2}{r} = \left(\frac{2\pi r}{T} \right)^2 \div r = \frac{4\pi^2 r}{T^2}$$

But Kepler's Third Law showed that $T^2 = kr^3$ for the planetary orbits where k was a constant. The expression kr^3 could therefore be substituted for T^2 on the bottom line of this relationship, to give:

$$a = \frac{4\pi^2 r}{kr^3} = \frac{4\pi^2}{kr^2}$$

Thus Kepler's Third Law amounted to the statement that the planetary accelerations decreased with distance from the Sun inversely as the square of the distance of the planet from the Sun, i.e. a planet three times further from the Sun than another would have $(1/3)^2$ or 1/9 of the other's acceleration. In fact the planetary orbits were not circles, but ellipses, so that both v and r varied around the orbit. An exact mathematical treatment was obviously complicated, but Hooke, Wren and Halley wondered if in fact all of Kepler's Laws might be reduced to a single inverse-square law of gravitational acceleration.

Hooke speculated along these lines in letters which he wrote to Isaac Newton in 1679. The circumstances of these letters will be described below. The next discussion of the problem appears to have been in January 1684 when Halley, then a man of twenty-eight, by his own account

> "came one Wednesday to town, where I met with Sr. Christ. Wren and Mr. Hook, and falling in discourse about it, Mr. Hook affirmed that upon that principle (of the inverse square law) all the Laws of the celestial motions were to be demonstrated, and that he himself had done it; I declared the ill success of my attempts; and Sir Christopher, to encourage the Inquiry, said that he would give Mr. Hook or me 2 months time to bring him a convincing demonstration thereof, and besides the honour, he of us that did it should have from him a present of a book of 40s. Mr. Hook then said that he had it, but that he would conceale it for some time, that others triing and failing might know how to value it when he should make it publick, however I remember Sir Christopher was little satisfied that he could do it, and tho Mr. Hook then promised to show it him, I do not yet find that in that particular he has been as good as his word".

In August 1684 Halley visited Isaac Newton in Cambridge and asked the then-famous mathematician what would be the form of the orbit for a body moving under an acceleration which varied inversely as the square of the distance from a second body. Newton had immediately answered "an ellipse", but could not produce a proof on the spot. In November 1684 Newton lectured on these matters in the University of Cambridge and Edmond Halley pressed him to make his knowledge of the subject more widely known. Thus it was that Sir Christopher Wren's challenge was answered neither by Hooke nor by Halley, but by Newton. And not with a few pages of mathematical calculations, but with what came to be regarded as the most important scientific book of any age--Newton's "Principia".

4. Isaac Newton

Newton (Figure 6) was born on Christmas Day, 1642, on a farm near the village of Colsterworth in Lincolnshire, England. While at school in nearby Grantham he proved to be a moderately able but exceedingly absent-minded scholar, more suited to an academic career than to farming. He went to Trinity College, Cambridge in 1661 as a "sub-sizar" or poor scholar who paid his way by doing odd jobs and waiting upon his tutor, who was Isaac Barrow, then Professor of Mathematics at the University. Newton took his degree in 1665 without particular distinction, and was then forced to leave Cambridge until the Fall of 1667 because the University closed for fear of spreading the Plague, which caused widespread death in London in the Summer of 1665. Newton spent these years on the farm at which he was born; it was during this period of solitary contemplation that he, by own own account, laid the foundations for the great accomplishments in mathematics, gravitation and optics for which he is now famous. We say "by his own account", for there is little concrete evidence of just what he did do at this time, and Newton described what had occupied him during these years only in 1716, when he was 73, and this after other scientists, including Hooke, had earlier disputed Newton's priority in making some discoveries.

What is clear is that the Newton who returned to Cambridge in 1667 had acquired an intellectual power that impressed those around him very greatly. In 1669 Isaac Barrow resigned his prestigious Professorship in favour of Newton, who thus became at the age of twenty-eight one of the most senior mathematicians in England. At this time his work was most visibly associated with the field of optics. He invented and constructed the first reflecting telescope (Figure 7), whose design eliminated the false-colour effects associated with the lensed (refracting) telescopes of his day. The invention brought much interest from the Royal Society,

Isaac Newton (1642-1727)

Fig. 6

and in 1672 Newton presented his first scientific paper to the Society, a discussion of the nature of light and colour based on his experiments and explaining the superior performance of his telescopes. The paper was revolutionary in its time and was vigorously criticised, mostly incorrectly, by numerous members of the Society. Newton's reaction to the criticism is exemplified by his subsequent letter to the German mathematician Gottfried Leibniz, in which he writes:

"I was so persecuted with discussions arising from the publication of my theory of light that I blamed my own imprudence for parting with so substantial a blessing as my quiet to run after a shadow".

Newton avoided the Royal Society as much as possible for the next years--his "quiet" served him better than discussions with its members, whose persistent questions, relayed to him by the Society's Secretary (who seems to have wished to set Newton at loggerheads with Hooke) did not help Newton advance his own thinking. Newton was capable of immense concentration in the solitude of his rooms in Trinity College, often forgetting to eat, or, on going out, neglecting to arrive at the destination he had in mind when he left. He was essentially a recluse whose own mental capacity served him better than the interaction with his contemporaries.

In 1679 Hooke became Secretary of the Royal Soceity and attempted to persuade Newton to renew contact with his fellow scientists and to pass opinion on some ideas which he (Hooke) had published in 1674. These ideas were that a planet would move in a straight line in the absence of any force, so that what kept a planet in orbit must be a force directed <u>towards the Sun</u>, that force decreasing inversely as the square of the distance from the Sun. Hooke was not, it will be noted, restating Kepler; Kepler had been quite Aristotelian at this point and had presumed that a force <u>along the orbit</u> was needed to propel a planet round the Sun. Huygens' formula for the centripetal acceleration had been published in 1673, and Hooke's remarks were likely inspired by that--possibly by a calculation from Kepler's Third Law similar to the one we made in the previous section. Again by his own account (to Halley in 1686) but not publicly, Newton went to work on the theory of the planetary motions and solved the problem of showing that <u>all</u> of Kepler's Laws could be explained from such an inverse-square law of attraction. However, it was not until 1684, when Halley went to Cambridge to discuss the matter with him, that Newton intimated to anyone that he, at last, had wrestled the 2000-year-old problem of the planetary motions to the ground.

To modern ears it seems incredible that any scientist could solve such a fundamental problem and keep the answer to himself, for the free and speedy communication of ideas is almost universal today in the scientific community. Newton's later claim that he had dealt with the solution in outline during the Plague years implies that he kept his knowledge to himself for nearly twenty years while others such as Huygens, Hooke and Halley struggled on in public. But Newton was both reclusive and absent-minded as well as being a formidable mathematician; his ability to make great strides in theoretical and practical science without need for contact with his contemporaries is demonstrated by what he did publish, and he may gave given little consideration to the interest others would have had in knowing of his achievements sooner.

Halley's visit persuaded him to write down what he knew of planetary motion and of gravity however, and this Newton did in his "Principia", sending the manuscript of the first of its three "books" to the Royal Society in April 1686. Hooke felt that he deserved some acknowledgement for having put the problem to Newton in the way he had in his letters of 1679. Newton claimed to have understood the inverse-square law before Hooke had asked him for his opinion on it. The dispute might have become very acrimonious had Halley not mediated it, to some extent out of fear that the later "books" would not see the light of day if Newton were too offended. In any event, in the printed version of the treatise, Newton acknowledges that Wren, Hooke and Halley had suggested the inverse-square law before the publication of his own ideas.

As published, the "Philosophiae Naturalis Principia Mathematica--Mathematical Principles of Natural Philosophy", usually referred to as the "Principia", presents in the first "book" a discussion of motions in empty space, in the second a treatment of motions in circumstances where a material medium is involved--including fundamentals of fluid flow around moving objects (applicable to ship hull and even (today) aircraft design) and of wave motions, and in the third an analysis of the structure of the Universe. The scope of the third book was, and is, breathtaking; in it Newton discussed the motions of Jupiter's satellites of the Moon, and of the planets in their elliptical orbits, then showed how to calculate the masses of the Sun and planets, estimated the average density of the Earth, calculated the details of the nonspherical shape of the Earth, explained the precession of the poles, discussed the disturbing effect of the Sun on the Moon's orbit, explained the tides, the orbits of the comets and certain phenomena of pendulums. The "Principia" completely dominated the science of motion for the next two hundred years, and still provides the fundamental basis for teaching that branch of science in today's universities. We will now examine some of the basic concepts