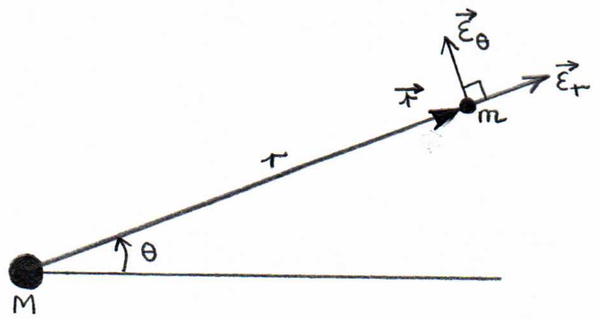
FULL THEORY OF TWO-BODY ORBITS IN A NEWTONIAN GRAVITATIONAL FIELD.

Consider a particle of small mass m which is attracted by a particle of large mass M. The force of gravitational attraction between these masses is along the line joining them. The acceleration of M will be small compared with that of m, so for a first approximation consider M to be at rest with m in motion around it.

There is no force between m and M out of the plane containing the line joining them, so the motion of m about M must be confined to this plane. Consider the origin of polar co-ordinates (r,Θ) to be taken at M and the position of m to be given by the vector $\vec{r}$ joining M to m and making an angle Θ with some fixed direction, as in the figure.



The co-ordinates r,Θ are the polar co-ordinates of m with respect to M. By Newton's Law of gravitation the force $\vec{F}$ exerted by M on m is

$$\vec{F} = -\frac{GMm}{r^2}\vec{\varepsilon}_r$$

where $\vec{\varepsilon}_r$ is the unit vector in the direction of $\vec{r}$, namely from M to m. ($\vec{\varepsilon}_\Theta$ shown in the diagram is the unit vector perpendicular to $\vec{\varepsilon}_r$ in the direction of increasing angle Θ). In terms of (r,Θ) co-ordinates and their time derivatives, the acceleration of a particle moving on a general curve in a plane is

$$\vec{a} = (\ddot{r}-r\dot{\Theta}^2)\vec{\varepsilon}_r + (r\ddot{\Theta} + 2\dot{r}\dot{\Theta})\vec{\varepsilon}_\Theta$$

so that using $\vec{F} = m\vec{a}$ we have for the equation of motion

$$-\frac{GMm}{r^2}\vec{\varepsilon}_r = m(\ddot{r}-r\dot{\theta}^2)\vec{\varepsilon}_r + M(r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{\varepsilon}_\theta$$

As $\vec{\varepsilon}_r$ and $\vec{\varepsilon}_\theta$ are perpendicular vectors, this vector equation of motion can be expressed as two independent simultaneous equations by equating the coefficients first of $\vec{\varepsilon}_r$ and then of $\vec{\varepsilon}_\theta$ on the two sides of the equation.

i.e. $\frac{-GMm}{r^2} = m(\ddot{r}-r\dot{\theta}^2)$ [1] [clearly m will cancel out of both of these]

and $0 = m(r\ddot{\theta} - 2\dot{r}\dot{\theta})$ [2]

Equation [2] will hold for any law of force giving no force component in the $\vec{\varepsilon}_\theta$ - direction - i.e. for any central force, whether inverse-square or not.

Now $r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r} \times \frac{d}{dt}\{r^2\dot{\theta}\}$

$\therefore \frac{d}{dt}\{r^2\dot{\theta}\} = 0$, i.e. $(r^2\dot{\theta})$ is constant throughout the motion.

$mr^2\dot{\theta}$ is the angular momentum of m about the axis through M perpendicular to the plane of the motion. Thus angular momentum is conserved throughout the motion. You should be able to show that the area ΔA swept out in time Δt by the vector r is equal to $\frac{1}{2}r^2\Delta\theta$, so that $\frac{1}{2}r^2\dot{\theta}$ = rate of area sweep $\dot{A}$. This conservation law thus also implies Kepler's Second Law, that the rate of area sweep is constant through the motion, i.e. that equal areas are swept out in equal times. This conservation law will evidently hold for any central force.

To find the equation of the path on which m moves, it is necessary to eliminate the time from equation [1]. This can be achieved by taking the variable $u = \frac{1}{r}$, and calculating the derivatives $\frac{du}{d\theta}$ and $\frac{d^2u}{d\theta^2}$ in terms of $\dot{r}$, $\ddot{r}$ and $\dot{\theta}$.

Doing this, and using $r^2\dot{\theta}$ = constant, k, from [2], we can reduce [1] to

$$\frac{d^2u}{d\Theta^2} + u = \frac{GM}{k^2}$$

A solution of this equation, which can be verified by substitution, is

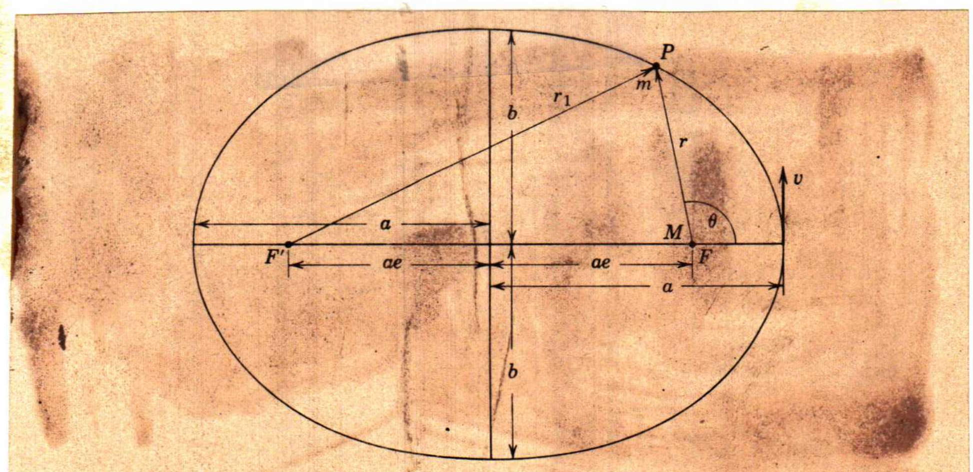
$$u = \frac{1}{r} = C\cos(\Theta+\alpha) + \frac{GM}{k^2}$$

where C and α are arbitrary constants of integration. The constant α can be made equal to zero by suitable choice of the fixed axis from which Θ is measured, so that the equation of the path of m around M is

$$\frac{1}{r} = C\cos\Theta + \frac{GM}{k^2}$$

This is the general equation for a conic section--an ellipse (special case a circle), a parabola, or a hyperbola.

The ellipse (circle) is the only closed curve among these, so it follows that if m is to have a closed orbit around M, the path must be elliptical, or in special circumstances, circular.



An ellipse, $r_1 + r = 2a$.

Consider the properties of an ellipse (see figure) with the two foci F and F'. The defining property of an ellipse is that the sum of the distances from the foci to a point on the ellipse is the same for all points. If r_1 and r are these distances, then $r_1 + r$ is equal to a constant which can be seen from the figure to be equal to 2a, the length of the major axis. The distance between the foci is less than 2a, and equals 2ae where e is defined to be the eccentricity ($e=0 \rightarrow$ a circle).

From the cosine law in the triangle FPF',

$$r_1^2 = r^2 + (2ae)^2 + 2(2ae)r\cos\Theta$$

Also $r_1 + r = 2a$. Eliminating r_1, you will find

$$a(1-e^2) = r(1 + e\cos\Theta), \text{ which can be written}$$

$$\frac{1}{r} = \frac{e\cos\Theta}{a(1-e^2)} + \frac{1}{a(1-e^2)}$$

Thus m moves on an ellipse around M as a focus, and with

$$a = \frac{k^2}{GM(1-e^2)} \qquad \text{and } e = \frac{Ck^2}{GM}$$

The area of an ellipse is πab, and the major and minor axes are in the ratio $1 : \sqrt{1-e^2}$ (see figure). If T is the period of revolution of m about M, then the area πab is swept out in time T. The rate of area sweep is thus

$$\dot{A} = \frac{\pi ab}{T} \text{ also } = \tfrac{1}{2}r^2\dot{\Theta} = \frac{k}{2}$$

$$T = \frac{2\pi ab}{k} = \frac{2\pi a^2\sqrt{1-e^2}}{k} = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

Thus the square of the period, T^2, is proportional to a^3 (Kepler's Third Law).

TWO-BODY ORBITS (continued)

The work done by the gravitational attraction of a mass M on a mass m which falls towards it from a distance $r = \infty$ to $r = R$, is

$$\Delta W = \frac{GMm}{R}$$

This can be considered as a release of an amount ΔV of gravitational "potential energy" V, where $\Delta V = \frac{GMm}{R}$. If we define the gravitational potential energy of the mass m to be zero when it is infinitely distant from M, then its "gravitational potential energy" when at a distance r is

$$V = -\frac{GMm}{r}$$

The minus sign indicates that gravitational energy is released when the mass m falls from infinity to a distance r from M. The fall may be thought of as converting gravitational potential energy into kinetic energy T of the falling mass m, so that T + V = constant.

If T = 0 and V = 0 at $r = \infty$, then at any other distance r, T + V = 0

$$\text{i.e.} \quad \frac{1}{2}mv^2 - \frac{GMm}{r} = 0.$$

which is the relation we derived before for v as a function of r.

In the case of an elliptic or circular orbit, a similar energy equation holds; in this case

$$\tfrac{1}{2}mv^2 - \frac{GMm}{r} = \text{[a negative constant.]}$$

As r varies, so does the velocity v. In an elliptical orbit there is a continuous interchange between the kinetic energy of the motion of the mass m and its gravitational potential energy in the presence of the mass M. It can be shown from this that at a point in the orbit where m is at a distance r from M, the velocity is given by:-

$$v^2 = 2GM\left(\frac{1}{r} - \frac{1}{2a}\right)$$

REFINEMENTS

1. Allowance for the motion of M.

If M is comparable with m (as in the case of orbits of the components of double stars about each other) it is unreasonable to assume that M remains fixed while m revolves around it. The acceleration of m at any moment is equal to $\frac{GM}{r^2}$ towards M. The acceleration of M is equal to $\frac{Gm}{r^2}$ towards m. The acceleration of m relative to M is therefore

$$\frac{GM}{r^2} + \frac{Gm}{r^2} = \frac{G(M+m)}{r^2} \quad \text{towards M.}$$

With this refinement, Kepler's Third Law becomes

$$T = \frac{2\pi a^{3/2}}{\sqrt{G(M+m)}}$$

for the orbit of m relative to M considered as fixed. This modification is extremely important in analysis of the orbits of double stars.

2. General Relativity.

If we describe the gravitational attraction of M for m using the theory of general relativity formulated by Einstein, the equation at the top of page

3 becomes

$$\frac{d^2u}{d\Theta^2} + u = \frac{GM}{k^2} + \frac{3GM}{c^2} u^2$$

The introduction of the relativity term $\frac{3GM}{c^2}u^2$ causes the direction of the major axis of the elliptical orbit to change with time.

The angle it turns through during one revolution of m is

$$\Delta\Theta = \frac{6\pi GM}{c^2a(1-e^2)}$$

Within the solar system, this effect has been investigated for the planet Mercury. This is the closest planet to the Sun and has therefore the smallest value of a; the predicted rotation of the major axis is 43" arc/century. Mercury also has a fairly eccentric orbit (e = 0.2056) which makes the change in direction of the major axis relatively easy to determine (in comparison with that of other planets). Complications are introduced by the gravitational attraction of the other planets, and by the fact that the Sun is not quite spherical. These effects cause a rotation of the major axis of the orbit more than 10 times that predicted by general relativity, but can be allowed for with sufficient accuracy to permit an approximate check. There has been controversy in recent years over the effect of the non-sphericity of the Sun, but it appears that the prediction of general relativity is approximately correct.