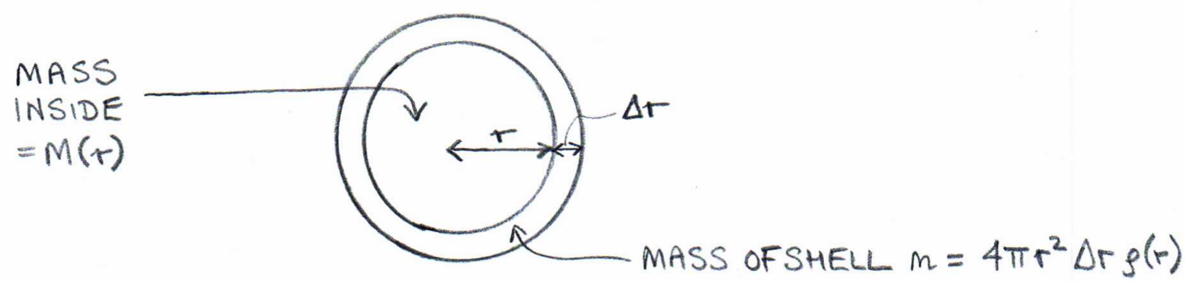
Mathematical theory of stellar interiors

Consider an element of the outer layer of a star. The gravitational force on this element due to the attraction of the star as a whole acts inwards towards the centre of the star. The star therefore has a permanent tendency to collapse under its own gravitation which can only be balanced if the inwards-acting gravitational force on each element of the star is counter-acted by some other, outwards-acting, force. We know the surface temperatures, chemical compositions and pressures of stellar surfaces from their optical spectra, and therefore look for a suitable force which can be exerted by a gas which is mostly hydrogen. The most likely source of such a force is simple gas pressure, obeying $P = NkT$ where P is the pressure, N is the number of particles per unit volume, k is Boltzmann's constant and T is the temperature.

If the star is not to collapse, the gravitational pressure due to the weight of each layer of the star acting inwards must be compensated by an excess of gas pressure on the inside of the layer relative to the outside. For the whole star to be maintained in equilibrium, neither expanding nor contracting, the gas pressure must increase inwards through the star at exactly the rate required to balance the weight of the concentric layers of the star.

Let $M(r)$ be the mass within a radius r of the centre of the star. The inwards gravitational force on a mass m at distance r from the centre is

$$F = \frac{G\, m\, M(r)}{r^2}$$



Consider a spherical shell of stellar material, of radius r and thickness Δr. Its mass will be $m = 4\pi r^2 \Delta r . \rho(r)$ where $\rho(r)$ is the density at radius r. The inwards gravitational pressure due to gravitational attraction on this shell will be the force per unit area of the shell, i.e.

$$\Delta P_{grav} = \frac{F}{4\pi r^2} = \frac{G\ M(r)}{r^2}\ \rho(r)\ \Delta r$$

If the star is neither expanding nor contracting, $\Delta P_{grav} + \Delta P_{gas}$ must = 0 across the shell. Dropping the subscript 'gas' on the gas pressure, as it is the only pressure that will appear later, we have the first equation of stellar structure, the equation of mechanical equilibrium,

$$\frac{\Delta P}{\Delta r} = \frac{G\ M(r)\ \rho(r)}{r^2} \qquad [1]$$

Now M(r) and ρ(r) are not independent. Remember that M(r) stands for the mass <u>inside</u> radius r while ρ(r) is the density <u>at</u> radius r. Consider again the spherical shell of thickness Δr and radius r. The mass inside is M(r). M(r+Δr) is the mass inside plus the mass ΔM of the shell. This ΔM is none other than the mass m we used above, so we must have

$$\frac{\Delta M}{\Delta r} = 4\pi\ r^2\ \rho(r) \qquad [2]$$

We may re-write the equation for the gas pressure as

$$P = NkT = \frac{k\rho T}{\mu m_H} \qquad [3]$$

where ρ is the mass density, μ is the mean molecular weight of the stellar gas, and m_H is the mass of a hydrogen atom. Hydrogen molecules will have μ = 2, hydrogen atoms μ = 1, and a gas of protons and electrons (fully ionized hydrogen), μ = ½. We now have three equations in four unknowns, M(r), ρ(r) and the pressure P and temperature T (both functions of r). This set is therefore insufficient (we must have as many equations as unknowns if we are to be able to find a unique solution for the interior parameters of the star), and to proceed we have to involve some more physics. We have implied that the gas pressure P must decrease outwards from the centre of the star. The third equation tells us that this implies that the temperature T must increase outwards from the centre. This in turn implies that heat must be flowing outwards from the centre of the star. (Note that we here have the important result that a star must be luminous to avoid gravitational collapse.) The equations for heat flow will help to define the system further.

Heat flow can generally take place by any of three processes, namely conduction, convection and radiation. Conduction is unimportant in gases, even at the extremes of physical conditions encountered in stars - it will be neglected here. Convection is heat transfer by bulk circulatory motion of the matter in a region, such as the flow of warm air from the furnace to individual rooms in a centrally heated house. Convection occurs if large temperature gradients exist in the star. Consider a small volume of stellar material, initially in equilibrium inside a star. Imagine it to be displaced ouwards by a small distance, so that it rises into a cooler region where the gas pressure is less. It will expand adiabatically, satisfying $P = \text{const}.\rho^{\gamma}$, where γ is the constant ratio of the specific heats of the stellar material at constant pressure and at constant volume. It thus reduces its pressure and temperature by this expansion. If its pressure and temperature are reduced below that of the surrounding gas by this process, it will be forced back to its original position, which is therefore one of stable equilibrium. But if the surrounding temperature gradient in the stellar gas is great enough, the small volume of material may be hotter than its surroundings even after adiabatic expansion, and it will keep on rising. This is the unstable situation which will lead to the formation of convection currents in the star. Mathematically

$$\text{If } \frac{\Delta T}{\Delta r} > \left\{1 - \frac{1}{\gamma}\right\} \frac{T}{P} \frac{\Delta P}{\Delta r}, \text{ convection occurs}$$

$$\text{if } \frac{\Delta T}{\Delta r} < \left\{1 - \frac{1}{\gamma}\right\} \frac{T}{P} \frac{\Delta P}{\Delta r}, \text{ convection does not occur.}$$

If convection does not occur, heat flow will be by radiation. In practice we can usually assume the latter and inspect our solution for the variation of T and P within the star to see if the conditions for no convection are indeed satisfied. Anywhere they are not, we must substitute convection and the adiabatic equation for what follows. To deal with the equation for heat flow by radiation, we first introduce the quantity $L(r)$ to describe the total flow of radiation per unit time across the lower surface of the shell of radius r. L will be measured in ergs/second, and clearly $L(R^*)$ must = L^*, the observed stellar luminosity. The flow of radiant energy is impeded by its own inertia and by the opacity of the stellar material. The momentum of an energy E in the form of radiation is equal to its effective mass times its velocity c, or $\frac{E}{c}$. The radiation momentum crossing unit area of the star at radius r per unit time is thus $L(r)/4\pi r^2 c$. If κ is the opacity of the stellar material per unit mass then the optical depth of the shell is $\kappa\rho\Delta r$. Radiation theory tells us that the momentum of radiation crossing unit area per unit time, multiplied by the optical depth through a region, is equal to the change in the radiation pressure ΔP_{rad} across that region. Now $P_{rad} = \frac{1}{3}\sigma T^4$ where σ is Stefan's constant, so that

$$\Delta P_{rad} = \frac{4}{3}\sigma T^3 \Delta T$$

relates the change in radiation pressure across the shell to the change in temperature across it. We therefore have that

$$\frac{4}{3}\sigma T^3 \Delta T = -\frac{L(r)}{4\pi r^2 c} \kappa\rho\Delta r$$

$$\text{i.e.} \quad \boxed{\frac{\Delta T}{\Delta r} = -\frac{3\, L(r)\, \kappa\, \rho(r)}{16\pi\, r^2\, T^3\, \sigma\, c} \quad [4]}$$

where the density has been written as a variable with r to remind us that it will be different in different spherical shells. This equation tells us how the temperature change as we move outwards through the star is related to the luminosity $L(r)$ at the appropriate level in the star and to the opacity in that level. The minus sign reminds us that temperature will decrease with r if the energy flow $L(r)$ is positive outwards. For the moment the introduction of this equation does not save us, as we now have four equations in five unknowns, namely P, M, T, ρ and L, all as functions of r. We have also introduced the difficult problem of calculating κ for hydrogen-helium gas as a function of these, as well as that of calculating μ for different physical states of the gas. We can however define the system by looking at one more equation involving L. The luminosity must be generated by an energy supply somewhere, and further, L cannot increase across a spherical shell unless energy generation is going on within that shell. We denote by ε the rate of energy release per unit mass of hydrogen (ε is also calculable in principle, from the theories of thermonuclear energy release). Then if ΔL is the increase in L going from radius r to radius $r + \Delta r$,

$$\boxed{\Delta L = 4\pi\, r^2\, \Delta r\, \varepsilon\, \rho(r) \quad [5]}$$

This final equation gives us five equations for five unknowns, provided we know physics enough to be able to assign numbers to μ, κ and ε at different levels in the star. The set of equations can be 'solved' by starting with known values of P, T, M and L at the surface of the star ($r = R^*$) and working inwards across successive layers Δr thick, calculating μ, κ and ε each time for each new layer, and testing $\Delta T/\Delta r$ for radiative or convective heat transfer each time. This is all done on fast digital computers in fact, and the resulting solutions for all the parameters of the star as functions of r define different investigators' stellar 'models'. We can show how the mass-luminosity relation arises from these equations without getting involved in detailed computations however.

Suppose we label the central pressure and temperature of the star by P^o and T^o. We can make the rough approximations

$$\frac{\Delta P}{\Delta r} \sim \frac{P^o}{R^*} \quad \text{and} \quad \frac{\Delta T}{\Delta r} \sim \frac{T^o}{R^*}$$

at some intermediate value of r, given that P^o and T^o will be very much greater than the corresponding values at the surface of the star, so that $P^o - P^{surf} \sim P^o$ (where P^{surf} is the surface pressure) for example. If we consider conditions within a variety of different stars, the value of r at which these relationships are approximately correct will be a given fraction of the total radius of the star, R^*, in each case, so that we can put $r \propto R^*$ each time it appears in the equations.

Thus in equation [1], we must have

$$\frac{P^o}{R^*} \propto \frac{M^*}{{R^*}^2} \times \frac{P(r)}{T(r)}$$

But from equations [2] and [3] together we also have

$$\frac{M^*}{R^*} \propto {R^*}^2 \times \frac{P(r)}{T(r)}$$

In both of these relations we have used the fact that $M(r) \propto M^*$ if $r \propto R^*$. $P(r)$ and $T(r)$ stand for the pressure and temperature at the given radius r. This pair of equations enables us to eliminate their ratio $P(r)/T(r)$, and thus to discover the relationship

$$P^o \propto \frac{{M^*}^2}{{R^*}^4}$$

i.e. that the central pressure within a star will depend on the square of its mass and inversely on the fourth power of its radius. We can now use equation [3] to find the similar relationship for the central temperature T^o, because at the centre of the star equation [3] must hold with $P = P^o$

and $T = T^{o}$. The density which appears will be ρ^{o}, the central density. The assumptions we have already made about $M(r) \propto M^*$ and $r \propto R^*$ imply that $\rho^{o} \propto \bar{\rho}$, the mean density, and we know that

$$\bar{\rho} = \frac{M^*}{\frac{4}{3}\pi R^{*3}}$$

so that $$\rho^{o} \propto \frac{M^*}{R^{*3}}$$

Substituting this into equation [3] along with our relation for the pressure P^{o}, we find that

$$T^{o} \propto \frac{M^*}{R^*}$$

To derive the mass-luminosity relationship, we now substitute all of these into equation [4], so that we have

$$\frac{\Delta T}{\Delta r} \sim \frac{T^{o}}{R^*} \propto \frac{M^*}{R^{*2}} \text{ , also } \propto \frac{M^*}{R^{*3}} \times \frac{R^{*3}}{M^{*3}} \times \frac{1}{R^{*2}} \times L^*$$

The first proportionality comes from inspection on the left-hand side of equation [4], the second from the right-hand side. To be sure of this, try writing it out for yourself ! From inspection of these two proportionalities we can see how L* must itself depend on the other quantities. The dependences on P^{o} and T^{o} have been eliminated by our method of analysis, and the dependence on R* cancels out right at the end. We are left with

$$\boxed{L^* \propto M^{*3}}$$

which is not a bad approximation to the mass-luminosity law considering the crudity of the approach ! We have not even used the properties of hydrogen to find this (although we have assumed that stars of similar chemical composition will have interior structures such that $r \propto R^*$ - i.e. we have assumed that stars of different masses but the same chemical composition will be 'scaled replicas' of each other). What we have discovered is essentially this : our notions of the physics of the interior of a star, as expressed by the equations we have derived, are capable of producing the result that a star's luminosity should, under plausible assumptions, depend on its mass only, and as the third power of that mass. This is a very satisfactory first test of the plausibility of our theory. The rest is very complex astrophysics - which has been done fully and properly with the aid of large modern computers.