July 12, 1980

Relativistic Jets:

1. The basic equations have been written in a useful and exact form in Series S-(B)-p6. These are useful by being exactly analogous to the equations used in the CH paper.

2. Consider now the homologous solution:

We proceed for simplicity with $B_z = B_\varphi = B_\varpi = 0$

which is accomplished by taking $B_z = 0$.

Then

(i) radial equation of motion may be written: (S(B)-5 eq. 5)

$$\rho u_\varpi \frac{\partial}{\partial \varpi}(\mathcal{S} u_\varpi) + \rho u_z \frac{\partial}{\partial z}(\mathcal{S} u_\varpi) = \frac{\rho \mathcal{S} u_\varphi^2}{\varpi} - \frac{\partial p}{\partial \varpi}$$

We try homologous forms:

$\rho, \mathcal{S}$ are functions of z only; $u_z = w_z(z)$, $u_\varpi = w_\varpi \frac{\varpi}{R}$

$u_\varphi = w_\varphi \, \varpi/R$; $w_\varphi(z)$, $w_\varpi(z)$

$p = p_i - (p_i - p_e)\left(\frac{\varpi}{R}\right)^2$: p_i, p_e are functions of z only

$\frac{dR}{dz} = \frac{w_\varpi}{w_z}$ as usual : Note that $w \equiv (w_\varpi, w_\varphi, w_z)$ is now the 3 part of the 4 vector velocity at $\varpi = R$

Then: Radial equation becomes exactly:

$$\boxed{w_z \frac{d}{dz}(\rho w_\varpi) = \frac{\rho w_\varphi^2}{R} + \frac{2(p_j - p_e)}{\rho R}}$$ ... (A)

and (ii) from 5 (B) - 6

$$w_\varphi = \Omega R \cdot \frac{L}{\rho \Omega \varpi^2}$$

where
$$\left\{ \begin{array}{l} L = \Omega R_s^2 \left(\frac{\varpi}{R}\right)^2 \\ \Omega = \text{constant} \end{array} \right\}$$
Σ ∴ both L, Ω are properly constant on stream lines

$$\boxed{\rho w_\varphi = \Omega R \left(\frac{R_s}{R}\right)^2}$$ --- (B)

The continuity equation becomes (see series 5 (C)) exactly (still treating γ as $\gamma(\varpi, z)$)

$$\boxed{R^2 \rho w_z = \text{constant}}$$ --- (C)

finally, (iv), the energy equation 5 (B) -7 is

$$\rho \gamma = \text{constant}$$

(see series (C) for discussion in differential form, where w_ϖ there may be considered as $\sqrt{w_\varpi^2 + w_\varphi^2}$ to include w_φ)

but

$$\gamma^2 = 1 + w_z^2 + (w_\varpi^2 + w_\varphi^2)\frac{\varpi^2}{R^2}$$

∴ as $\rho \equiv \rho(z)$ our homologous form fails at this point:

either: (i) $\frac{w_\varpi^2 + w_\varphi^2}{w_z^2} << 1$ so that $\boxed{\gamma^2 \simeq 1 + w_z^2}$ ... (D-1)

This is lowest order result

or

(ii) set $\gamma = \gamma(\varpi = R(z))$ which maximizes the effects of W_ϖ, W_ϕ ...

$\therefore \boxed{\gamma_s^2 \gamma^2 = 1 + (W_z^2 + W_\phi^2 + W_\varpi^2) w_{zs}^2}$ — (D-ii)

or (iii) use $\boxed{\gamma_s^2 \gamma^2 = \langle \gamma_s^2 \gamma^2 \rangle_\varpi = 1 + \left(W_z^2 + \frac{1}{3}(W_\phi^2 + W_\varpi^2)\right) w_{zs}^2}$ — (D-iii)

"Consider the analysis of the simplest case:

– also discussed in series (C) and (D)

We scale as usual by sonic point quantities:

$$\tau \equiv p_j/\rho \,/\, p_{js}/\rho_s = (\rho/\rho_s)^{a_s^2} \equiv \mathcal{P}/\mathcal{P}_s$$

$$\therefore \quad \tau \equiv \mathcal{P} = \rho^{a_s^2} \qquad \text{in sonic units}$$

$$`\gamma' \equiv \gamma/\gamma_s \; : \; R \equiv R/R_s \; , \; z = z/R_s$$

$$w_{zs} \equiv \gamma_s a_s c \; , \; \mathcal{P}_s = (1+a_s^2)\frac{\epsilon_s}{\rho_s c^2} \; ; \; p_{js}/\rho_s = \frac{a_s^2 \mathcal{P}_s}{1+a_s^2}$$

$$C_e = \frac{p_{es}}{p_{js}}$$

Then we have the beautifully simple relations:

Continuity $\quad \underline{R^2 \rho w_z = 1}$ -- (1)

Energy $\quad \underline{\gamma \tau = 1}$ -- (2)

equation of state $\quad \underline{\tau = \rho^{a_s^2} = (R^2 w_z)^{-a_s^2}}$ -- (3)

dual equation of motion $\quad w_z \frac{d}{dz}\left(\tau w_z \frac{dR}{dz}\right) = \left(\frac{2}{1+a_s^2}\right)\frac{1}{\gamma_s^2}\frac{1}{R}\left\{\tau - C_e f(z)\, \tau^{-1/a_s^2}\right\} \pm \frac{w_\varphi^2}{R}$ -- (4)

boundary stream line $\quad w_\varpi = w_z \frac{dR}{dz}$ -- (5)

on boundary $\quad \gamma_s^2 \gamma^2 = 1 + w_z^2 + w_\varphi^2 + w_\varpi^2 \quad$ at $\varpi = R(z)$ -- (6)

polar momentum $\quad \tau w_\varphi = \left(\frac{\Omega R_s}{w_{zs}}\right)\frac{1}{\mathcal{P}_s}\frac{1}{R}$ -- (7)

These equations can be and shall be treated numerically

The equations to be solved are

(E-1) $$w_z \frac{d}{dz}\left(\tau w_z \frac{dR}{dz}\right) = \frac{2}{(1+a_s^2)} \frac{1}{\gamma_s^2} \frac{1}{R} \left\{ \tau - C_e f(z)\, \tau^{-1/a_s^2} \right\} + \frac{\tau w_\varphi^2}{R}$$

$$\tau = (R^2 w_z)^{-a_s^2}$$

$$\tau w_\varphi = \left(\frac{\Omega R_s}{w_{zs}}\right) \frac{1}{\rho_s} \frac{1}{R}$$

plus an equation relating w_z and R from (2), (3) and (6)

Thus:

$$\gamma^2 = (R^2 w_z)^{2a_s^2} \qquad \text{from (2) and (3)}$$

and $$\gamma^2 \gamma_s^2 = 1 + \left\{ w_z^2 \left(1 + \left(\frac{dR}{dz}\right)^2\right) + \left(\frac{\Omega R_s}{w_{zs}\rho_s}\right)^2 R^{-2} (R^2 w_z)^{2a_s^2} \right\} w_{zs}^2$$

∴

(E-2) $$\boxed{(R^2 w_z)^{2a_s^2} \gamma_s^2 = 1 + w_{zs}^2 w_z^2 \left(1 + \left(\frac{dR}{dz}\right)^2\right) + \left(\frac{\Omega R_s}{\rho_s}\right)^2 \frac{(R^2 w_z)^{2a_s^2}}{R^2}}$$

∴ E-1 and E-2 must be solved together

Note: at the sonic point

$$\gamma_s^2 = 1 + \gamma_s^2 a_s^2 + \left(\frac{\Omega R_s}{\rho_s}\right)^2$$

$$\boxed{\gamma_s^2 = \frac{1 + \left(\frac{\Omega R_s}{\rho_s}\right)^2}{1 - a_s^2}}$$

when

(i) $\Omega = 0$

(ii) $(dR/dz)^2 \ll 1$

Then:

$$\boxed{W_z \frac{d}{dz}\left(\tau W_z \frac{dR}{dz}\right) = \left(\frac{2}{1+a_s^2}\right)\frac{1}{\gamma_s^2}\frac{1}{R}\left\{\tau - C_{eff}(\tau)\,\tau^{-1/a_s^2}\right\}}$$

$$\boxed{\tau = (R^2 W_z)^{-a_s^2}}$$

$$\boxed{W_z^2 - \frac{R^{4a_s^2}}{a_s^2} W_z^{2a_s^2} + \frac{1}{\gamma_s^2 a_s^2} = 0}$$

$$\underline{\gamma_s^2 = \frac{1}{1-a_s^2}}$$

$a_s^2 = 1/3$ * — * we must handle a better equation of state* – thermal plus non-thermal plus coupling

$\therefore \quad \gamma_s^2 = 3/2$

$$W_z^2 - 3R^{4/3} W_z^{2/3} + 2 = 0$$

let $x = W_z^{2/3}$

$$\underline{x^3 - 3R^{4/3}x + 2 = 0}$$

now

$$\Delta = -4\cdot 27\cdot R^4 + 4\times 27 = \underline{108(1-R^4)} \leq 0$$

$\therefore \quad R = 1$

$$x^3 - 3x + 2 = 0$$

$$(x-1)(x^2 + x - 2) = 0$$

$\therefore$ double root at $x = +1$ $\quad x_1, x_2 = +1 \quad \underline{x_3 = \frac{-1 - \sqrt{1+8}}{2} = -2}$

$\therefore$ when $\underline{R > 1}$, $\Delta > 0$ irreducible case

$$\tan\phi = -\frac{\sqrt{108(R^4-1)}}{\sqrt{108}} = -\sqrt{R^4-1}$$

ϕ in 2nd quadrant

$$\therefore \begin{cases} x_1 = 2R^{2/3}\cos\phi/3 \\ x_2 = -2R^{2/3}\cos(60^\circ - \phi/3) \\ x_3 = -2R^{2/3}\cos(60^\circ + \phi/3) \end{cases}$$

< 0

roots of bifurcating solution

x_1, x_3 give two branches which ~~reconverge~~ diverge at large R when $\phi \to \frac{\pi}{2}^+$

$x_1 \to$ supersonic
$x_3 \to$ Subsonic

$$\sec^2\phi = R^4$$

$$\cos^2\phi = 1/R^4 \qquad \cos\phi = -1/R^2$$

$$\sin\phi = \sqrt{1 - 1/R^4}$$

$$\cos^3\frac{\phi}{3} = \frac{3}{4}\cos\frac{\phi}{3} + \frac{1}{4}\cos\phi$$

irreducible case

letter:

$$A = -1 + i\sqrt{R^4-1} = a + ib$$

$$B = -1 - i\sqrt{R^4-1} = a - ib$$

$$x_1 \sim \sqrt{3}\, R^{2/3} \qquad x_1^3 \sim 3\sqrt{3}\, R^2$$

clearly better than 10% when R is 2

$$x \sim \sqrt{3}\, R^{2/3}$$

$$\underline{\underline{w_2 \sim 3^{3/4} R}}$$

Hence:

$$\tau \sim \left(3^{3/4}\right)^{-1/2} R^{-1}$$

$$\underline{\underline{\tau \sim \frac{3^{-1/4}}{R}}} \qquad : \quad \underline{\underline{W_2 \sim 3^{3/4} R}}$$

$$\underline{\underline{\gamma \sim 3^{3/4} R}}$$

and

$$3^{3/4} R \frac{d}{dz}\left(3^{1/2} \frac{dR}{dz}\right) = \frac{1}{R}\left\{ \frac{3^{-1/4}}{R} - C_e f(z)\, 3^{3/4} R^3 \right\}$$

$R \gg 1$

$$R \frac{d^2R}{dz^2} = \frac{1}{3^{1/2} R^2}\left\{ 1 - 3C_e f(z) R^4 \right\}$$

$$\frac{d^2R}{dz^2} = \frac{1}{3^{1/2}} \frac{1}{R^3}\left\{ 1 - 3C_e f(z) R^4 \right\}$$

$C_e = 1$ here

$$\therefore \boxed{\frac{d^2R}{dz^2} \simeq \frac{1}{3^{1/2}} \frac{1}{R^3}\left\{ 1 - 3f(z) R^4 \right\}}$$

$\underline{R >\sim 3\text{-}4}$

free expansion

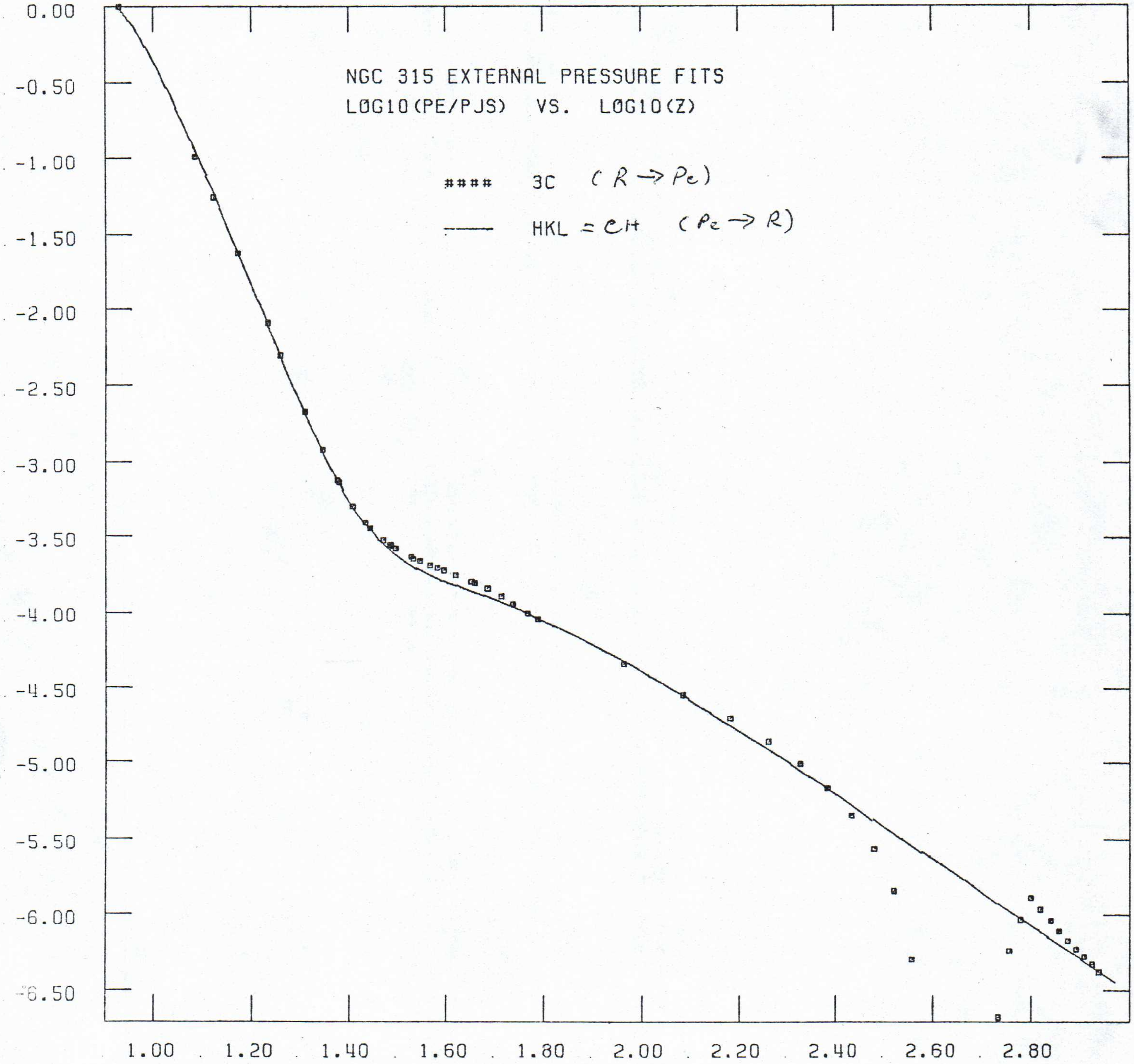
$$\frac{1}{2}\frac{d}{dR}\left(\frac{dR}{dz}\right)^2 = \frac{1}{3^{1/2}} \frac{1}{R^3}$$

$$\frac{1}{2}\left(\frac{dR}{dz}\right)^2 \simeq -\frac{1}{2 \cdot 3^{1/2} R^2} + \text{const}$$

$\therefore$ either free expansion at constant cone angle.

but also possibly nearly zero.

depends critically on $f(z)$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3C (R → Pe)
—— HKL = CH (Pe → R)
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-1.00
-1.50
-2.00
-2.50
-3.00
-3.50
-4.00
-4.50
-5.00
-5.50
-6.00
-6.50
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1.60
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2.00
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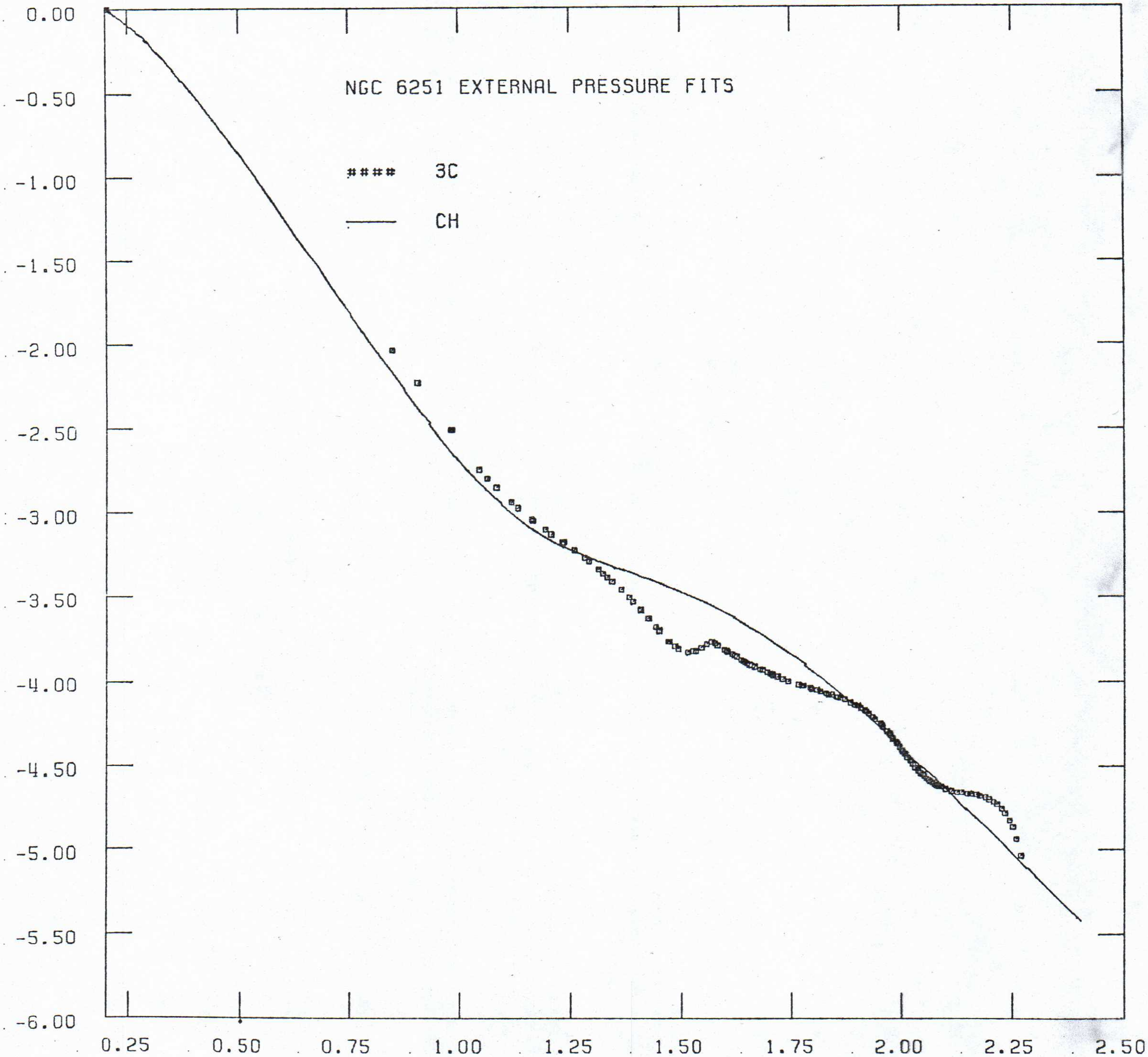
2.40
2.60
2.80


NGC 6251 EXTERNAL PRESSURE FITS
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-4.00
-4.50
-5.00
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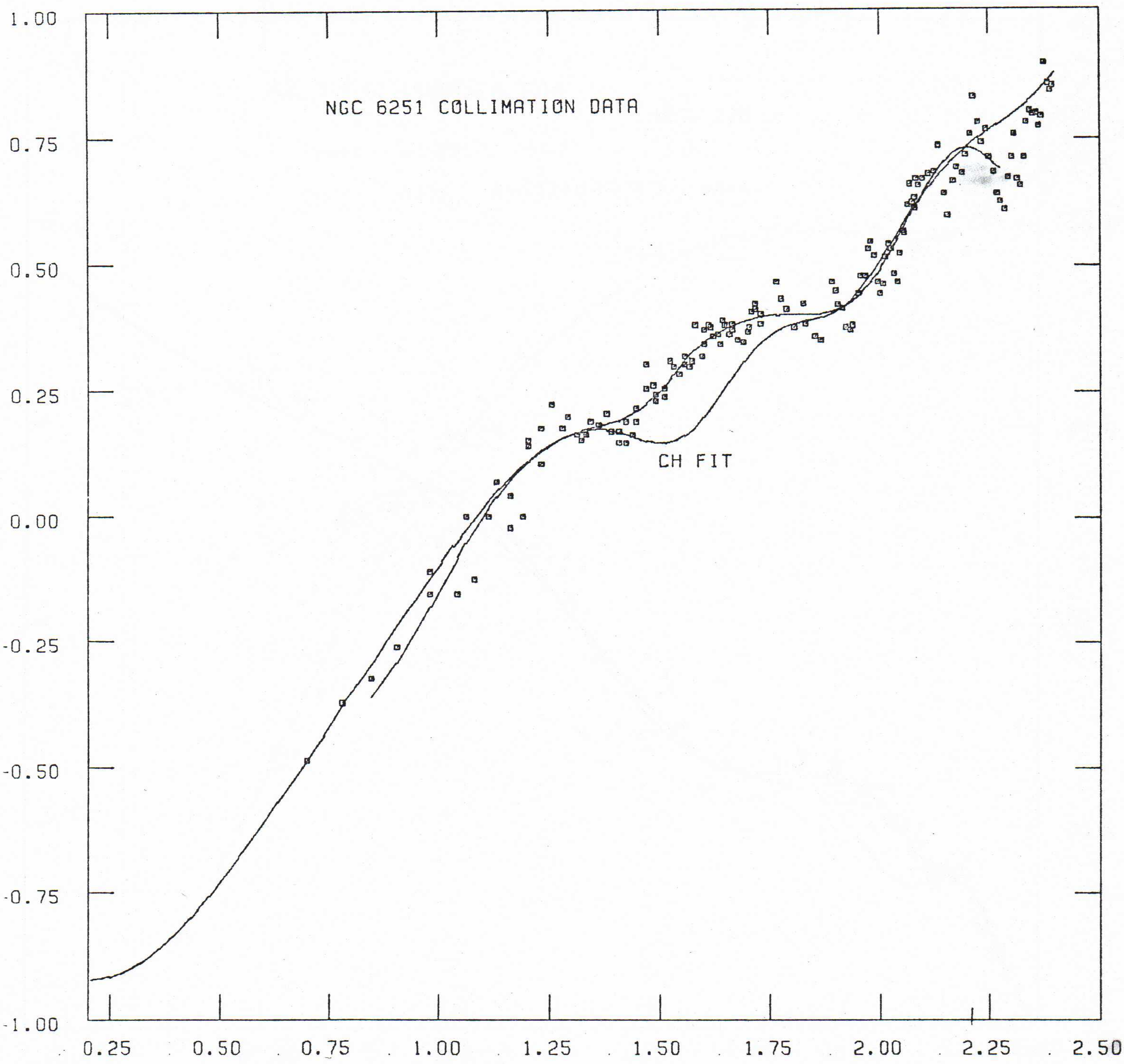

NGC 6251 COLLIMATION DATA
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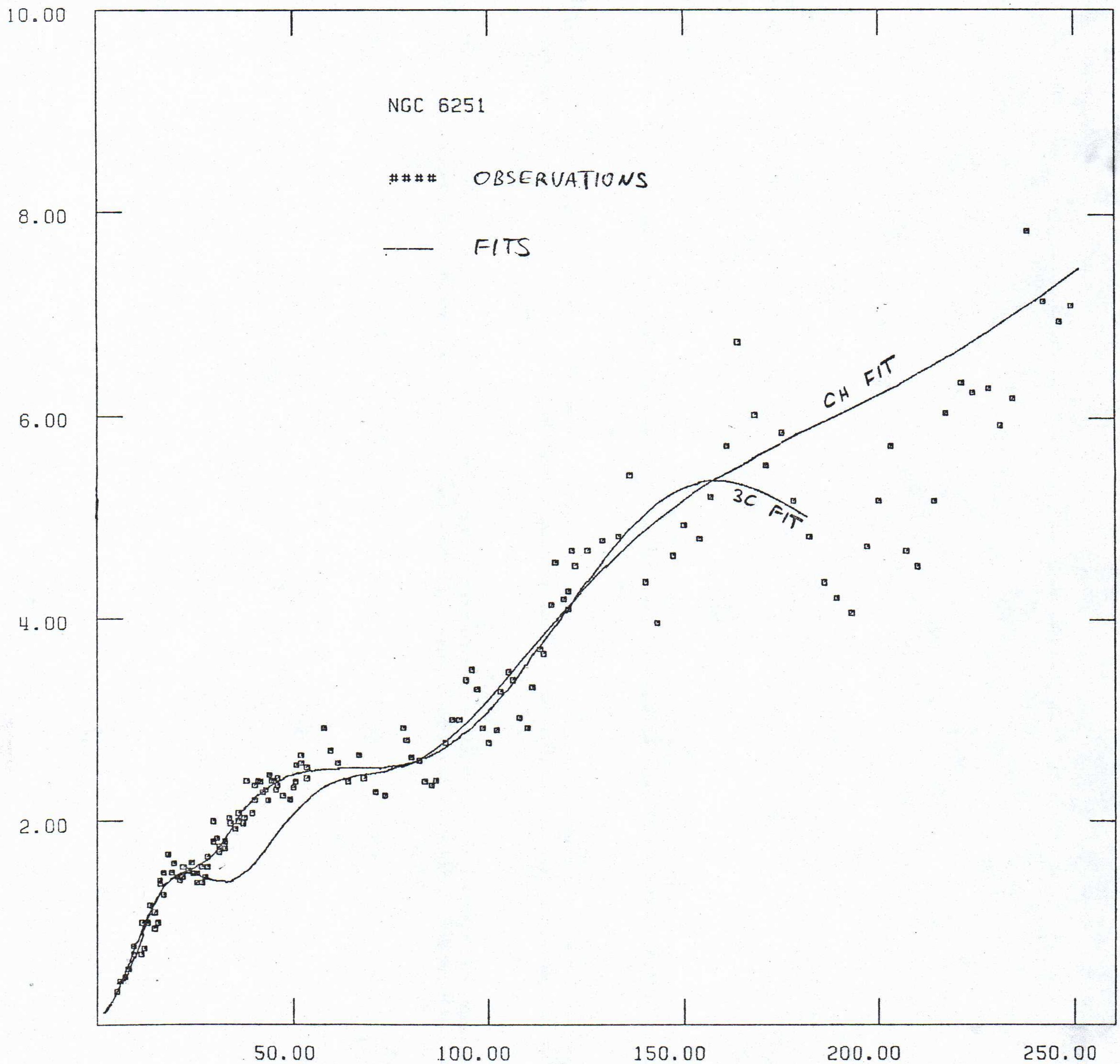

NGC 6251
OBSERVATIONS
—— FITS
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3C FIT
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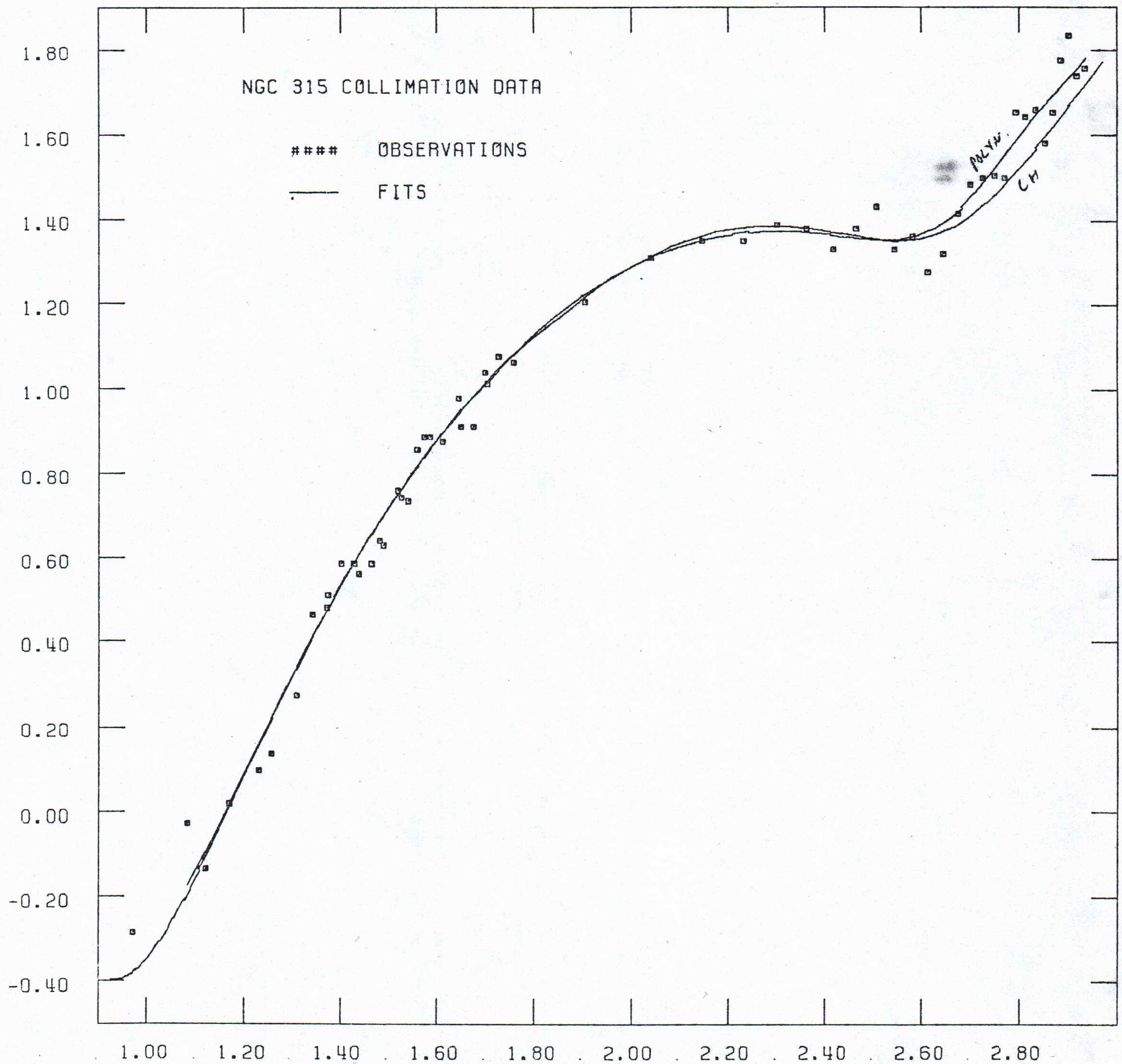

NGC 315 COLLIMATION DATA
OBSERVATIONS
FITS
POLYN.
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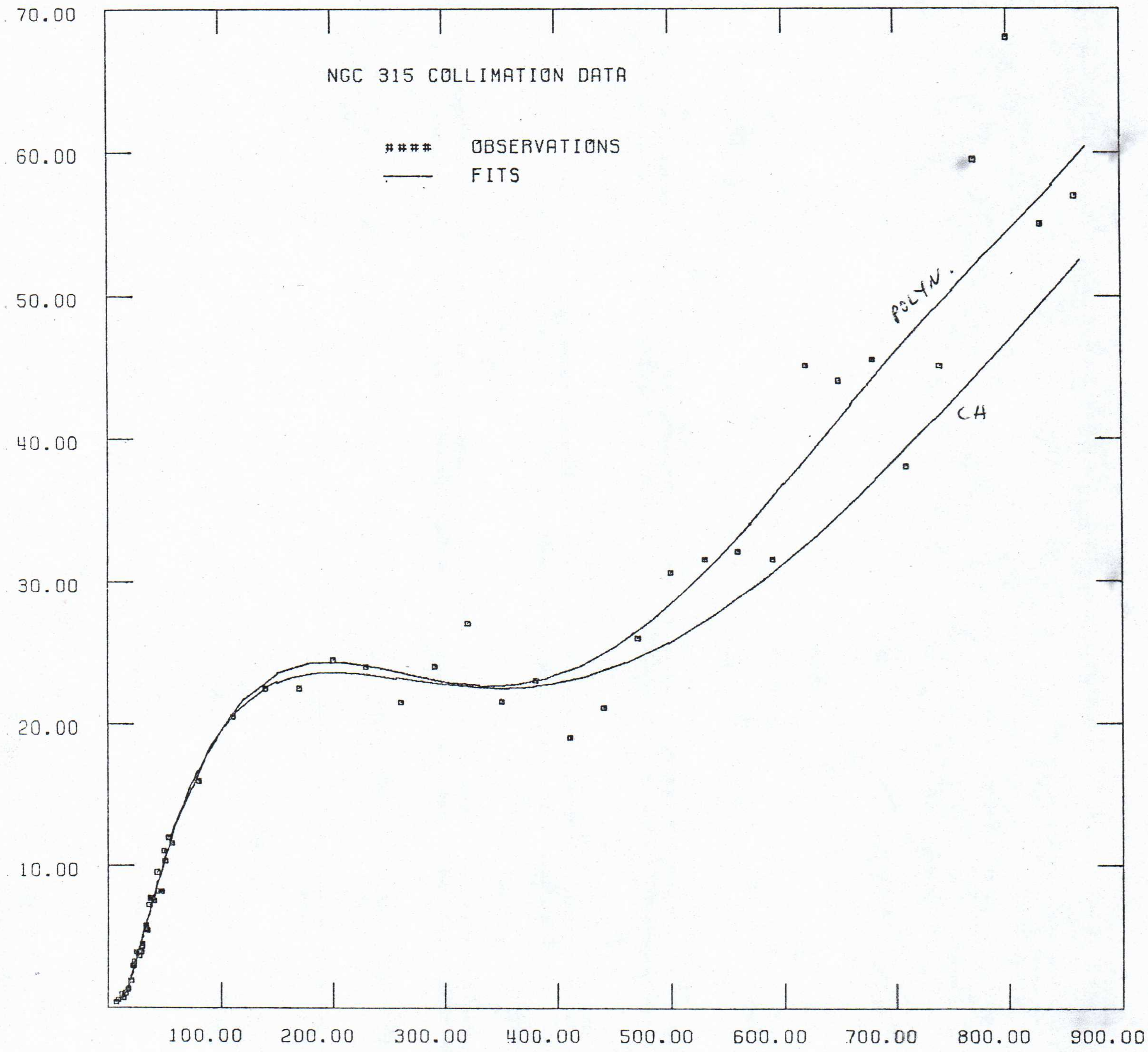
NGC 315 COLLIMATION DATA
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500.00
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800.00
900.00