QUEEN'S UNIVERSITY AT KINGSTON

School of Graduate Studies and Research

December 13, 1982

PHYSICS 831*

THREE HOURS

Answer both Questions from Section A and ONE Question from EACH of Sections B and C, for a total of FOUR Questions. All Questions carry equal marks.

Chemical Rubber Co. Mathematical Tables are available on request.

The following constants and formulae may be assumed in any Question:

$\mu_0 = 1.26 \times 10^{-6}$ m.kg.C
$\varepsilon_0 = 8.85 \times 10^{-12}$ $\mathrm{Nt^{-1}.m^{-2}.C}$
$c = 2.998 \times 10^{8}$ $\mathrm{m.s^{-1}}$
$e = 1.60 \times 10^{-19}$ C (charge of the electron)
$m = 9.11 \times 10^{-31}$ kg (mass of the electron)

Fields of an accelerated moving charge

$$\underset{\sim}{E} = \frac{q}{4\pi\varepsilon_0[1-\underset{\sim}{\beta}.\underset{\sim}{n}]^3}\left[\frac{\underset{\sim}{n}-\underset{\sim}{\beta}}{\gamma^2 R^2} + \frac{\underset{\sim}{n}\times\{(\underset{\sim}{n}-\underset{\sim}{\beta})\times\dot{\underset{\sim}{\beta}}\}}{cR}\right], \quad \underset{\sim}{B} = \frac{[\underset{\sim}{n}]\times\underset{\sim}{E}}{c}$$

Larmor Formula

$$\langle P\rangle = \frac{q^2[\dot{\beta}^2]}{6\pi\varepsilon_0 c}$$

Classical radiation reaction force

$$\underset{\sim}{F}_r = \frac{q^2\ddot{\underset{\sim}{v}}}{6\pi\varepsilon_0 c^3}$$

Multipole expansion of vector potential from system of charges (radiative terms only)

$$\underset{\sim}{a}(\underset{\sim}{r}_f,\omega) = \frac{\mu_0}{4\pi}\cdot\frac{e^{ikR_0}}{R_0}\left[\dot{\underset{\sim}{d}} - ik\left(\tfrac{1}{6}\dot{\underset{\sim}{D}} + \underset{\sim}{\mathcal{M}}\times\underset{\sim}{n}\right) + \ldots\right]$$

$$\underset{\sim}{d} = \sum_i q_i \underset{\sim}{r}_i$$

$$\underset{\sim}{\mathcal{M}} = \tfrac{1}{2}\sum_i q_i(\underset{\sim}{r}_i\times\underset{\sim}{v}_i)$$

$$D_\alpha = \sum_\beta n_\beta\Big(\sum_i q_i(3x_{i\alpha}x_{i\beta} - r_i^2\delta_{\alpha\beta})\Big)$$

Kirchhof-Helmholtz Integral

$$\psi_P = \frac{1}{4\pi}\oiint_S\left(\psi\underset{\sim}{\nabla}\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r}\underset{\sim}{\nabla}\psi\right).\,d\underset{\sim}{S}$$

....continued

Fourier Transform Integrals $$F(t) = \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega , \quad f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{i\omega t} dt$$

Parseval's Theorem for Fourier Transforms $$\int_{-\infty}^{\infty} F^2(t)\, dt = 4\pi \int_0^{\infty} |f(\omega)|^2 d\omega$$

$$\int_0^{\pi/2} \sin^3\theta \, d\theta = \frac{2}{3} \qquad \sum_{n=0}^{n=N-1} x^n = \frac{x^N - 1}{x - 1} \qquad \int_0^a \frac{x^{2m+1}}{\sqrt{a^2 - x^2}} dx = \frac{m!\, 2^m\, a^{2m+1}}{(1;2;m+1)}$$

where the notation

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a}.\underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a}.\underset{\sim}{b})\underset{\sim}{c} \qquad (\alpha;\beta;\gamma) = \alpha(\alpha+\beta)(\alpha+2\beta)\ldots\ldots(\alpha+(\gamma-1)\beta)$$

In spherical polar co-ordinares (r, θ, ϕ),

$$\underset{\sim}{\nabla} \times \underset{\sim}{v} = \frac{\hat{\underset{\sim}{r}}}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\phi \sin\theta) - \frac{\partial v_\theta}{\partial\phi}\right\} + \frac{\hat{\underset{\sim}{\theta}}}{r}\left\{\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right\} + \frac{\hat{\underset{\sim}{\phi}}}{r}\left\{\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right\}$$

where $\underset{\sim}{v}$ is any vector $(v_r \hat{\underset{\sim}{r}}, v_\theta \hat{\underset{\sim}{\theta}}, v_\phi \hat{\underset{\sim}{\phi}})$

Section A. Answer BOTH Questions

1. Calculate the average power radiated per unit solid angle by an electric dipole of moment $\underline{d}$ rotating in a plane with constant angular velocity Ω, as a function of the angle from the axis perpendicular to the plane. Average over the period of the rotation of the dipole. Describe the polarization state of the radiation as a function of the angle.

2. A bound electron in an atom is modelled as a classical harmonic oscillator of natural (undamped) oscillation frequency ω_0, subject to damping by the radiation reaction. Derive expressions for:

(a) the band width between half-power frequencies of the power spectrum radiated by the electron when set into free (unforced) oscillation,

(b) the cross-section of the electron for scattering of plane waves of frequency ω, integrated over all scattering directions.

State carefully any assumptions made.

.....continued

Section B. Answer ONE Question from this Section

3. A one-dimensional diffraction grating produces amplitude modulation

$$\psi = 1 + m\cos(2\pi x/d) \qquad (m<1)$$

and extends from $-D/2 < x < +D/2$ along the x axis. The width D of the aperture is large compared with the wavelength d of the amplitude modulation. Derive an expression for its Fraunhofer diffraction pattern when it is illuminated with plane waves of wavelength λ.

Describe how this pattern would be modified if the transparency of the aperture also varied linearly from unity at its centre to zero at its edge, producing the illumination:

$$\psi = (1 + m\cos(2\pi x/d))(1 - 2|x|/D)$$

Note: your results may be expressed as the convolutions of simple functions, without actual evaluation of convolution integrals.

4. Derive an expression for the far-field intensity distribution of the radiation diffracted by a linear array of N identical transparent square apertures illuminated in phase by a monochromatic plane wave of constant amplitude over the array. The apertures are each of side a and their centres are distance D apart. The apertures do not overlap one another, are all in the same plane, and the area of the plane between the apertures is completely opaque.

What restrictions would you place on the validity of the expression you have derived, and why ?

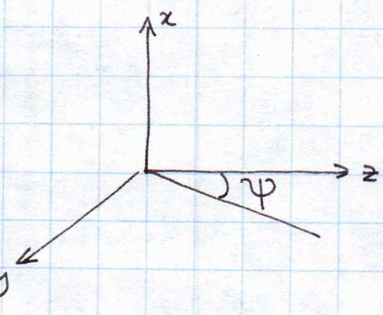
Section C. Answer ONE Question from this Section

5. Two identical charges +q are constrained to oscillate harmonically at angular frequency ω about the origin of coordinates along the z axis. Their motions have the same amplitude a but are exactly in antiphase, so that they reach opposite ends of their travel together and pass through the origin, in opposite directions, together. Calculate the angular distribution of their radiation, neglecting their mutual Coulombic interaction. Assume their motions to be exactly collinear, i.e. ignore the practical complication of how they pass each other at the origin.

6. Two particles with masses m_1 and m_2, carrying positive charges q_1 and q_2, start directly towards each other with relative velocity v_0 ($\ll c$) at infinity. They are prevented from colliding "head on" by their Coulomb repulsion, but their motions are collinear at all times. Calculate the total energy radiated during their encounter, stating any assumptions made.

1. Find the expressions for the radiation from an electric dipole d rotating in a plane with constant angular velocity Ω as a function of the angle Θ from the axis perpendicular to the plane, averaged over the period of the rotation. Describe the polarization state of the radiation as a function of this angle.

Here
$$d_x = d\cos\Omega t \qquad \ddot{d}_x = -\Omega^2 d\cos\Omega t$$
$$d_y = d\sin\Omega t \qquad \ddot{d}_y = -\Omega^2 d\sin\Omega t$$
$$d_z = 0$$



$$\dot{\underset{\sim}{d}} = \underset{\sim}{\Omega}\times\underset{\sim}{d}$$
$$\ddot{\underset{\sim}{d}} = \underset{\sim}{\Omega}\times\dot{\underset{\sim}{d}} \qquad (\Omega = \text{const})$$
$$= \underset{\sim}{\Omega}\times\underset{\sim}{\Omega}\times\underset{\sim}{d} \quad = -\Omega^2\underset{\sim}{d}$$

$$\underset{\sim}{E}_{rad} = \frac{\underset{\sim}{n}\times(\underset{\sim}{n}\times\ddot{\underset{\sim}{d}})}{4\pi\epsilon_0 c^2 R}$$

$$\underset{\sim}{n} = -\underset{\sim}{i}\sin\Psi + \underset{\sim}{k}\cos\Psi$$
$$\ddot{\underset{\sim}{d}} = -\Omega^2 d\,(\underset{\sim}{i}\cos\Omega t + \underset{\sim}{j}\sin\Omega t)$$

$$\underset{\sim}{n}\times\ddot{\underset{\sim}{d}} = -\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -\sin\Psi & 0 & \cos\Psi \\ \cos\Omega t & \sin\Omega t & 0 \end{vmatrix}\Omega^2 d = -\Omega^2 d\,(-\underset{\sim}{i}\cos\Psi\sin\Omega t + \underset{\sim}{j}\cos\Psi\cos\Omega t - \underset{\sim}{k}\sin\Psi\sin\Omega t)$$

$$\underset{\sim}{n}\times\underset{\sim}{n}\times\ddot{\underset{\sim}{d}} = -\Omega^2 d\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -\sin\Psi & 0 & \cos\Psi \\ -\cos\Psi\sin\Omega t & \cos\Psi\cos\Omega t & -\sin\Psi\sin\Omega t \end{vmatrix}$$

$$= -\Omega^2 d\left(-\underset{\sim}{i}\cos^2\Psi\cos\Omega t - \underset{\sim}{j}(\cos^2\Psi + \sin^2\Psi)\sin\Omega t - \underset{\sim}{k}\sin\Psi\cos\Psi\cos\Omega t\right)$$

$$|E_{rad}| = \frac{\Omega^2 d}{4\pi\epsilon_0 c^2 R}\sqrt{(\cos^4\Psi + \sin^2\Psi\cos^2\Psi)\cos^2\Omega t + \sin^2\Omega t}$$

$$|H_{rad}| = \frac{1}{\mu_0}\underset{\sim}{B}_{rad} = \frac{\underset{\sim}{n}\times\underset{\sim}{E}}{\mu_0 c}$$

$$R^2|\underset{\sim}{E}\times\underset{\sim}{H}| = \frac{\Omega^4 d^2 \cdot R^2}{16\pi^2\epsilon_0 c^4 R^2 \mu_0 c}\left(\cos^2\Psi\cos^2\Omega t + \sin^2\Omega t\right)$$

$$\left\langle\frac{dP(\Psi)}{d\Omega}\right\rangle = \frac{\Omega^4 d^2}{32\pi^2\epsilon_0 c^3}(1 + \cos^2\Psi)$$

$\Psi = \Theta$ of Question

averaged over one period

E_{rad} at $\psi = 0$, $\sin\psi = 0$
$\cos\psi = 1$

$E_{rad} \propto -\underset{\sim}{i}\cos\Omega t - \underset{\sim}{j}\sin\Omega t$ circular polarized.

E_{rad} at $\psi = 90°$, $\sin\psi = 1$
$\cos\psi = 0$

$E_{rad} \propto -\underset{\sim}{j}\sin\Omega t$ linear polarized.

General polarization state is <u>elliptical</u>.

2. (a) Eqn. of motion of electron in unforced oscillation

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = 0 \qquad F_r = \frac{q^2\dddot{\underset{\sim}{v}}}{6\pi\epsilon_0 c^3} \rightarrow \Gamma \sim \frac{q^2\omega_0^2}{6\pi\epsilon_0 c^3 m}$$

so $\Gamma = 6{\cdot}25\times10^{-24}\,\omega_0^2$

Let
$$\left.\begin{aligned} x &= Ae^{\alpha t} \\ \dot{x} &= A\alpha e^{\alpha t} \\ \ddot{x} &= A\alpha^2 e^{\alpha t}\end{aligned}\right\} \qquad \alpha^2 + \Gamma\alpha + \omega_0^2 = 0$$

$$\alpha = \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\omega_0^2}}{2}$$

$$x(t) = Ae^{-\Gamma t/2}e^{\pm i\omega_0 t} \quad (t>0) \qquad \underline{\text{if}} \quad \Gamma^2 \ll 4\omega_0^2$$

(ok for $\omega_0 \ll 10^{23}$ rad/s)

Then
$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{i\omega t}\,dt$$

$$= \frac{A}{2\pi}\int_0^{\infty} e^{\pm i\omega_0 t}e^{-\Gamma t/2}e^{i\omega t}\,dt$$

$$= \frac{A}{2\pi}\left\{\frac{1}{i(\omega\pm\omega_0) - \Gamma/2}\right\}$$

$$\ddot{x}(\omega) = \frac{A}{2\pi}\left\{\frac{\omega^2}{i(\omega\pm\omega_0) - \Gamma/2}\right\} = -\frac{A}{2\pi}\left\{\frac{\omega^2\left((\omega-\omega_0)i - \Gamma/2\right)}{(\omega-\omega_0)^2 + \Gamma^2/4}\right\}$$

$$|\ddot{x}| = \frac{A}{2\pi}\,\omega^2 \Big/ \sqrt{(\omega-\omega_0)^2 + \Gamma^2/4}$$

Larmor $\rightarrow$ $$P = \frac{q^2\ddot{x}^2}{6\pi\epsilon_0 c^3}$$

$$P(\omega) \doteq \frac{q^2A^2}{6\pi\epsilon_0 c^3 \cdot 4\pi^2}\left\{\frac{\omega_0^4}{(\omega-\omega_0)^2 + \Gamma^2/4}\right\} \quad (\omega\sim\omega_0)$$

$P \rightarrow \frac{1}{2}P_{max}$ at $\Delta\omega = \Gamma/2$. full width $2\Delta\omega = \Gamma = \underline{\dfrac{q^2\omega_0^2}{6\pi\epsilon_0 c^3 m}}$

(b) forced oscillation $\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = \frac{qE}{m} e^{-i\omega t}$

By analogy $x(t) = \frac{qE}{m} \cdot \frac{1}{-i\omega\Gamma + (\omega_0^2 - \omega^2)} e^{-i\omega t}$

So $x(\omega) = \frac{qE}{m} \cdot \frac{1}{(\omega_0^2 - \omega^2) - i\omega\Gamma}$ by inspection

$$\ddot{x}(\omega) = \frac{-qE\omega^2}{m((\omega_0^2-\omega^2) - i\omega\Gamma)}$$

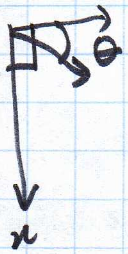
Hence by Larmor $P = \frac{q^2 \ddot{x}^2}{6\pi\epsilon_0 c^3} \rightarrow \frac{q^4 \omega^4 E^2}{m^2((\omega_0^2-\omega^2)^2 + \omega^2\Gamma^2)} \cdot \frac{1}{6\pi\epsilon_0 c^3}$

$$\frac{\langle P\rangle}{I} = \sigma = \frac{q^4\omega^4 \langle E^2\rangle}{m^2((\omega_0^2-\omega^2)^2 + \omega^2\Gamma^2)} \cdot \frac{1}{6\pi\epsilon_0 c^3} \cdot \frac{1}{\epsilon_0 c \langle E^2\rangle}$$

i.e. $\sigma = \frac{q^4\omega^4}{6\pi\epsilon_0^2 m^2 c^4((\omega_0^2-\omega^2)^2 + \omega^2\Gamma^2)}$

3. (a) Take $\psi = 1 + m\cos\left(2\pi x/d\right)$

$$\mathcal{F}(\psi) = \mathcal{F}(1) + m\,\mathcal{F}\left(\cos\frac{2\pi x}{d}\right) \qquad \mathcal{F} = \text{F.T.}$$



$$\mathcal{F}(1) = \int_{-\infty}^{\infty} e^{-ik_x x}\,dx = 2\pi\delta(k_x) \qquad k_x = k\sin\theta$$

$$\mathcal{F}\left(\cos\frac{2\pi x}{d}\right) = \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{2\pi i x/d} + e^{-2\pi i x/d}\right)e^{-ik_x x}\,dx$$

$$= \pi\delta\left(k_x - \frac{2\pi}{d}\right) + \pi\delta\left(k_x + \frac{2\pi}{d}\right)$$

(b) Take
$$\psi' = 1 \qquad |x| < D/2$$
$$= 0 \qquad |x| > D/2$$

$$\mathcal{F}(\psi') = \int_{-D/2}^{D/2} e^{-ik_x x}\,dx$$

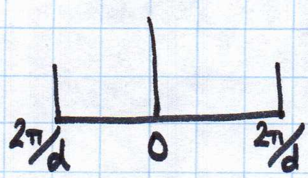
$$= -\frac{1}{ik_x}\left[e^{-ik_x D/2} - e^{ik_x D/2}\right]$$

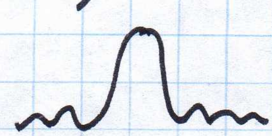
$$= 2\left(\frac{\sin k_x D/2}{k_x}\right)$$

Then for grating illumination = $\psi\psi'$ (multiplication)

F.T. $= \left(\mathcal{F}(1) + \mathcal{F}\left(\cos\frac{2\pi x}{d}\right)\right) \cap \mathcal{F}(\psi')$ using conv. theorem

$$= \left(2\pi\delta(k_x) + \pi\delta\left(k_x - \frac{2\pi}{d}\right) + \pi\delta\left(k_x + \frac{2\pi}{d}\right)\right) \cap 2\,\frac{\sin(k_x D/2)}{k_x}$$

= 

$\cap$ 

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Now add the taper

$$\psi'' = \left(1 - 2|x|/D\right)$$

This is the self-convolution of $\psi''' = 1 \quad |x| < D/4$

$= 0 \quad |x| > D/4$

$$\mathcal{F}(\psi''') = \frac{2\sin(k_x D/4)}{k_x}$$

$$\mathcal{F}(\psi'') = \mathcal{F}(\psi''') \times \mathcal{F}(\psi''') \quad \text{as} \quad \psi'' = \psi''' \cap \psi'''$$

So $\mathcal{F}(\psi'') = \dfrac{4\sin^2(k_x D/4)}{k_x^2}$

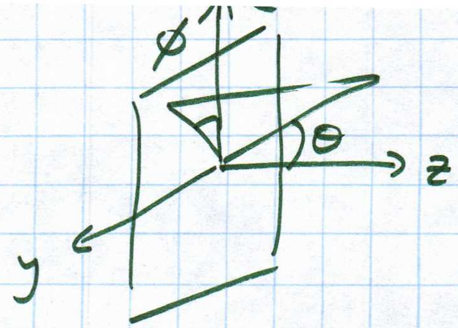
Now the new illumination is $\psi \times \psi' \times \psi''$

So the F.T. is $\mathcal{F}(\psi) \cap \mathcal{F}(\psi') \cap \mathcal{F}(\psi'')$

$$= \left(2\pi\delta(k_x) + \pi\delta\left(k_x - \frac{2\pi}{d}\right) + \pi\delta\left(k_x + \frac{2\pi}{d}\right)\right) \cap \frac{2\sin(k_x D/2)}{k_x} \cap \frac{4\sin^2(k_x D/4)}{k_x^2}$$

Power $\propto$ all of this squared.

4. Take single aperture, side a



$k_x = k \sin\theta \cos\phi$

$\psi = 1 \quad -a/2 \le x \le a/2$

$$\mathcal{F}(\psi) = \int_{-a/2}^{a/2} e^{-ik_x x}\, dx$$

$$= -\frac{1}{ik_x}\left[e^{-ik_x a/2} - e^{ik_x a/2}\right]$$

$$= 2\,\frac{\sin(k_x a/2)}{k_x}$$

Now consider the array layout function $\psi' = \sum_{n=0}^{N-1} \delta(x - nD)$

$$\mathcal{F}(\psi') = \int_{-\infty}^{\infty} \sum \delta(x-nD)\, e^{-ik_x x}\, dx$$

$$= \sum_{n=0}^{N-1} e^{-ik_x nD}$$

$$= \sum_{n=0}^{N-1} \rho^n \qquad \rho = e^{-ik_x D}$$

$$= \frac{\rho^N - 1}{\rho - 1} = \frac{e^{-ik_x ND} - 1}{e^{-ik_x D} - 1} = \frac{e^{ik_x ND/2} \cdot \sin(k_x ND)}{e^{ik_x D} \cdot \sin(k_x D)}$$

Illumination is $\psi \cap \psi'$

Fraunhofer Pattern $\propto |\mathcal{F}^2(\psi \cap \psi')| = |\mathcal{F}^2(\psi)| \cdot |\mathcal{F}^2(\psi')|$

$$\propto \frac{\sin^2(k_x a/2)}{k_x^2} \cdot \frac{\sin^2(k_x ND)}{\sin^2(k_x D)}$$

Restrictions

$$\psi_p = \frac{1}{4\pi}\oiint \left(\psi \underset{\sim}{\nabla}\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r}\underset{\sim}{\nabla}\psi\right)\cdot d\underset{\sim}{S} \qquad \begin{pmatrix}\text{Kirchof}\\ \text{Helmholtz}\end{pmatrix}$$

$$\downarrow$$

$$\iint_A \left(\psi \underset{\sim}{\nabla}\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r}\underset{\sim}{\nabla}\psi\right)\cdot d\underset{\sim}{S}$$

if ψ and $\underset{\sim}{\nabla}\psi$ $\to 0$ on transparent aperture

(1)

$$\underset{\sim}{\nabla}\left(\frac{e^{ikr}}{r}\right) = \frac{e^{ikr}}{r}\left(ik - \frac{1}{r}\right)\hat{\underset{\sim}{r}}$$

$$\nabla\underset{\sim}{\psi} = \hat{\underset{\sim}{k}} \qquad d\underset{\sim}{S} = \underset{\sim}{k}\,.\,dx\,dy$$

(2) Neglect $1/r$ term in $(ik - 1/r)$

(3) Neglect variation of $\hat{\underset{\sim}{r}}\cdot\hat{\underset{\sim}{k}}$, put $= \cos\theta$

(4) Neglect variation of $1/r$

Then $$\psi_p = \frac{-ike^{ikz}}{4\pi R}\iint e^{ik\Delta}(1+\cos\theta)\,dx\,dy$$

where $$r^2 = (x-x_A)^2 + (y-y_A)^2 + z^2$$
$$= z^2\left(1 + \left(\frac{x-x_A}{z}\right)^2 + \left(\frac{y-y_A}{z}\right)^2\right)$$
$$r \to z + \frac{(x-x_A)^2}{2z} + \frac{(y-y_A)^2}{2z}$$
$$\to z + \Delta$$

(5) Take out $(1+\cos\theta)$ obliquity (neglect its variation in integral)

$$\psi_p = \frac{-ike^{ikz}(1+\cos\theta)}{4\pi R}\iint e^{ik\left(\frac{(x-x_A)^2}{2z} + \frac{(y-y_A)^2}{2z}\right)}\,dx\,dy$$

This $\to$ a F.T. integral iff. neglect $\frac{x_A^2}{2z}$, $y_A^2/2z$, i.e. $z \gg \frac{N^2D^2}{\lambda}$

$\frac{x^2}{2z}$, $y^2/2z$, i.e. near axis.

5. This system has no net electric dipole or magn. dipole mt

Hence must calculate quadrupole

$$\underset{\sim}{a}(\underset{\sim}{r}_f,\omega) = \frac{\mu_0}{4\pi}\cdot\frac{e^{ikR_0}}{R_0}\cdot[-ik\dot{\underset{\sim}{D}}/6]$$

i.e. $$\underset{\sim}{B} = \underset{\sim}{\nabla}_f \times \frac{\mu_0}{4\pi}\cdot\frac{e^{ikr_f}}{r_f}\cdot(-ik)\cdot\frac{1}{6}\dot{\underset{\sim}{D}}$$

(cf. el. dipole $\quad -\dfrac{ik\dot{\underset{\sim}{D}}}{6} \longleftrightarrow \dot{\underset{\sim}{d}}$)

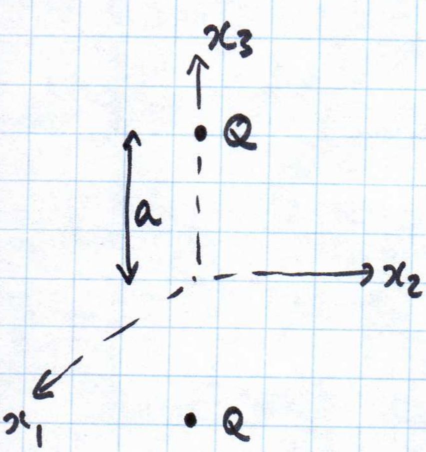
For radiative terms $$\underset{\sim}{\nabla}_f \equiv \frac{ike^{ikr_f}}{r_f}\underset{\sim}{n}$$

Hence $$\underset{\sim}{B}_{rad} = \frac{k^2\mu_0}{4\pi}\cdot\frac{1}{6}\cdot\frac{e^{ikr_f}}{r_f}\cdot(\underset{\sim}{n}\times\dot{\underset{\sim}{D}})$$

$$\left(\underset{\sim}{E}_{rad} = \left(\frac{c}{-ik}\right)\underset{\sim}{\nabla}_f\times\underset{\sim}{B} = -\frac{k^2}{4\pi\epsilon_0 c}\cdot\frac{1}{6}\cdot\frac{e^{ikr_f}}{r_f}(\underset{\sim}{n}\times(\underset{\sim}{n}\times\dot{\underset{\sim}{D}}))\right)$$

$$\frac{dP(\theta)}{d\Omega} = \frac{1}{2}\mu_0 c\,|H|^2 = \frac{1}{2}\mu_0 c\cdot\frac{k^4}{36\times16\pi^2}\cdot k^2c^2\,|\underset{\sim}{n}\times\underset{\sim}{D}|^2$$

$$= \frac{ck^6}{1152\pi^2\epsilon_0}\,|\underset{\sim}{n}\times\underset{\sim}{D}|^2$$



$$D_{\alpha\beta} = \sum_i(3x_\alpha x_\beta - r_i^2\delta_{\alpha\beta})q_i$$

$\alpha\neq\beta \quad D_{\alpha\beta} = 0$ as $x_1 = x_2 = 0$ always

$\alpha=\beta=1 \quad |D_{11}| = -a^2Q - a^2Q = -2a^2Q$

$2 \quad |D_{22}| = -a^2Q - a^2Q = -2a^2Q$

$3 \quad |D_{33}| = 3a^2Q - a^2Q + 3a^2Q - a^2Q = 4a^2Q$

So quadrupole tensor $\underset{\approx}{D} = 2Q\begin{pmatrix} -a^2 & 0 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & 2a^2 \end{pmatrix} e^{i\omega t}$

$$\underset{\sim}{D} = \underset{\approx}{D}\,\underset{\sim}{n} = \left(-2Qa^2 n_x \underset{\sim}{i} - 2Qa^2 n_y \underset{\sim}{j} + 4Qa^2 n_z \underset{\sim}{k}\right) e^{i\omega t}$$

Hence $|\underset{\sim}{n} \times \underset{\sim}{D}| = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ n_x & n_y & n_z \\ -2Qa^2 n_x & -2Qa^2 n_y & 4Qa^2 n_z \end{vmatrix}$

$$= 6Qa^2 n_y n_z \underset{\sim}{i} + 6Qa^2 n_x n_z \underset{\sim}{j}$$

$$|\underset{\sim}{n} \times \underset{\sim}{D}|^2 = (6Qa^2)^2 (n_y^2 n_z^2 + n_x^2 n_z^2)$$

$$= 36Q^2 a^4 n_z^2 (1 - n_z^2)$$

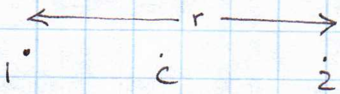
$n_z = \sin\theta$ where θ is the angle away from the x_3-axis

Hence $$\frac{dP(\theta)}{d\Omega} = \frac{ck^6}{1152\pi^2\epsilon_0} \cdot 36Q^2 a^4 \sin^2\theta \cos^2\theta$$

$$= \underline{\frac{ck^6 Q^2 a^4 \sin^2\theta\cos^2\theta}{32\pi^2\epsilon_0}}$$

6.

<u>Colliding particles.</u>



Dipole moment

$$\tau = r_1 + r_2$$

$$m_1 r_1 = m_2 r_2$$

$$r_1 = \frac{m_2 r}{m_1+m_2} \qquad r_2 = \frac{m_1 r}{m_1+m_2}$$

$$\underset{\sim}{d} = q_1 \underset{\sim}{r}_1 + q_2 \underset{\sim}{r}_2$$

$$= \left(\frac{q_1 m_2 - q_2 m_1}{m_1+m_2}\right)\underset{\sim}{r}$$

$$= \mu\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)\underset{\sim}{r} = \mu\,\Delta g\,\underset{\sim}{r}$$

Hence by Larmor's formula: $\langle P\rangle = \dfrac{\ddot{d}^2}{6\pi\epsilon_0 c^3} = \dfrac{\mu^2\Delta g^2\,\ddot{r}^2}{6\pi\epsilon_0 c^3}$ gives <u>dipole</u> radiation.

Under Coulomb interaction $\quad \mu\ddot{r} = +\dfrac{q_1 q_2}{4\pi\epsilon_0 r^2}, \quad \ddot{r} = +\alpha/\mu r^2$

$$\langle P\rangle = \frac{\mu^2\Delta g^2\alpha^2}{6\pi\epsilon_0 c^3\mu^2 r^4} = \frac{\Delta g^2\alpha^2}{6\pi\epsilon_0 c^3 r^4}$$

Then

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{dr}\cdot\frac{dr}{dt} = \dot{r}\frac{d\dot{r}}{dr}$$

$$\langle P \rangle = \frac{\Delta g^2 \alpha^2}{6\pi\epsilon_0 c^3 r^4}$$

Now – we want <u>total</u> radiation i.e. $\int \langle P \rangle dt = \int \frac{\langle P \rangle}{\dot{r}} \cdot dr$

Assume symmetric, i.e. $\int_{-\infty}^{0} = \int_{0}^{\infty}$. (neglect energy losses by radiation)

$$\ddot{r} = \frac{\alpha}{\mu r^2} = \dot{r}\frac{d\dot{r}}{dr}$$

$$\frac{1}{2}\dot{r}^2 = \int_{\infty}^{r} \ddot{r}\, dr + \frac{1}{2}v_0^2$$

$$\frac{1}{2}(\dot{r}^2 - v_0^2) = \int_{\infty}^{r} \frac{\alpha}{\mu r^2} dr = -\frac{\alpha}{\mu r}$$

$$\dot{r}^2 = v_0^2 - \frac{2\alpha}{\mu r}$$

$r = r_{min}$ when $\dot{r}^2 = 0$ $\quad v_0^2 = \frac{2\alpha}{\mu r_{min}}$, $\quad r_{min} = \frac{2\alpha}{\mu v_0^2}$

$$\Delta \mathcal{E} = 2\int_{\infty}^{r_{min}} \frac{\Delta g^2 \alpha^2}{6\pi\epsilon_0 c^3 r^4} \cdot \frac{1}{\sqrt{v_0^2 - \frac{2\alpha}{\mu r}}} dr$$

$$= \frac{2 \cdot \Delta g^2 \alpha^2}{6\pi\epsilon_0 c^3} \int_{\infty}^{r_{min}} \frac{1}{r^4} \cdot \frac{1}{\sqrt{v_0^2 - 2\alpha/\mu r}} dr$$

Let $x^2 = \frac{2\alpha}{\mu r}$ i.e. $r = \frac{2\alpha}{\mu x^2}$ $r^4 = \frac{16\alpha^4}{\mu^4 x^8}$ $\frac{1}{r^4} = \frac{\mu^4 x^8}{16\alpha^4}$

$$2x\,dx = -\frac{2\alpha}{\mu r^2}\,dr$$

$$\cancel{2}x\,dx = -\frac{\cancel{2}\alpha}{\mu}\cdot\frac{\mu^2 x^4}{4\alpha^2}\cdot dr$$

$$x^2_{max} = \frac{2\alpha}{\mu r_{min}} = \frac{2\alpha\,\mu v_0^2}{\mu . 2\alpha} = v_0^2$$

$$\frac{4\alpha^{\cancel{2}}\cancel{\mu}x\,dx}{\cancel{\alpha}\mu^{\cancel{2}}x^4} = -dr$$

$\int x^n \to \frac{1}{n+1}x^{n+1}$

x^2 $\frac{1}{3}x^3.$

$$-\frac{4\alpha}{\mu x^3}\,dx = dr$$

$$\text{So} \int = \int_{v_0}^{\infty} \frac{\mu^4 x^8}{16\alpha^4}\cdot\frac{4\alpha}{\mu x^3}\cdot\frac{1}{\sqrt{v_0^2 - x^2}}\,dx$$

$$= \int_{v_0}^{0} \frac{\mu^3}{4\alpha^3}\cdot\frac{x^5}{\sqrt{v_0^2-x^2}}\,dx$$

$$\boxed{\int_0^a \frac{x^5}{\sqrt{a^2-x^2}}\,dx = \frac{8a^5}{15}}$$

$$\frac{2\,\Delta g^2 \alpha^2}{6\pi\epsilon_0 c^3}\cdot\frac{\mu^3}{4\alpha^3}\cdot\boxed{\int_{v_0}^{0}\frac{x^5}{\sqrt{v_0^2-x^2}}\,dx}$$

$$\frac{2\,\Delta g^2\alpha^2\mu^3}{3\,\cancel{6}\pi\epsilon_0 c^3 . \cancel{4}\alpha^3}\cdot\boxed{\frac{\cancel{8}^{2}}{15}v_0^5} = \frac{2}{45}\frac{\Delta g^2\mu^3 v_0^5}{\pi\epsilon_0 c^3 \alpha}$$

$\alpha = \frac{q_1 q_2}{4\pi\epsilon_0}$

$$\frac{16}{\cancel{24}\times 15} = \frac{4}{45}.$$

$$= \frac{8}{45}\frac{\Delta g^2 \mu^3 v_0^5}{c^3}\left(\frac{1}{q_1 q_2}\right)$$

OK!

agrees with L. & L.