Chapter Two: Observable Properties of World-Models

CHAPTER TWO: Observable Properties of World-Models - Theory

2.1 The Red Shift Relation

In order to relate observable quantities to the functional form of the cosmological scale factor $R(t)$, we need to analyse the propagation of light through the model Universe. While the main relationships can be obtained within the neo-Newtonian framework, they can be found more readily from the GR formulation (which is better suited to the description of photon propagation). The starting-point is the fact that photons travel on null geodesics such that the events along the path of a photon through the model satisfy

$$ds^2 = 0$$

which for radially-travelling photons in the Robertson-Walker metric (~~1.9~~ 1.10) reduces to

$$c^2 dt^2 = R^2(t) \left\{ d\sigma^2/(1-k\sigma^2) \right\}$$

We will normally be interested in radially-travelling photons because we can choose either the observer or the emitter of the radiation to be at the origin of the (σ, θ, ϕ) coordinates. The above equation can be rewritten in the more useful form

$$(2.1) \quad cdt/R(t) = (+/-)d\sigma/\sqrt{(1-k\sigma^2)}$$

which relates the time interval dt to the radial coordinate interval $d\sigma$ along a section of a photon's path at coordinates (σ, t). The (+) sign will apply for photons which are travelling outwards from the origin of coordinates, the (-) sign for photons which are travelling inwards. If we integrate (2.1) along the finite path from a point of emission σ_E at emission time t_E to a point of observation σ_O at observation time t_O we find that

$$\int_{t_E}^{t_O} \frac{cdt}{R(t)} = \binom{+}{-} \int_{\sigma_E}^{\sigma_O} \frac{d\sigma}{\sqrt{(1-k\sigma^2)}}$$

Now consider applying this relation to the path of a photon emitted from a source at coordinate σ_E at time t_E and observed by an observer at the origin ($\sigma=0$) at the later time t_O. In this case we are dealing with an inwards-propagating photon, so the (-) sign applies in (2.1) and

$$\int_{t_E}^{t_O} \frac{cdt}{R(t)} = -\int_{\sigma_E}^{0} \frac{d\sigma}{\sqrt{(1-k\sigma^2)}}$$

The integral on the right is just $S_k(\sigma_E)$ from Section 1.2.5, which is a constant for a given source of radiation (remember that σ_E is the constant co-moving coordinate of a given FO, galaxy, etc.). This means that a second photon, emitted from the same source at time $(t_E+\Delta t_E)$ and received by the same observer at time $(t_O+\Delta t_O)$ will also satisfy

$$\int_{t_E+\Delta t_E}^{t_O+\Delta t_O} \frac{cdt}{R(t)} = -\int_{\sigma_E}^{0} \frac{d\sigma}{\sqrt{(1-k\sigma^2)}}$$

where the right hand side has the same value $S_k(\sigma_E)$. This requires that

$$\int_{t_E}^{t_O} \frac{cdt}{R(t)} = \int_{t_E+\Delta t_E}^{t_O+\Delta t_O} \frac{cdt}{R(t)}$$

$$= \int_{t_E}^{t_O} \frac{cdt}{R(t)} + \frac{c\Delta t_O}{R(t_O)} - \frac{c\Delta t_E}{R(t_E)}$$

which in turn means that

(2.2) $\Delta t_E/R(t_E) = \Delta t_O/R(t_O)$

Now if the source is radiating a monochromatic frequency ν_E, the number of wave crests emitted in time Δt_E will be $N = \nu_E \Delta t_E$. As this same number N of wave crests is received by the observer in the time Δt_O, the wave frequency perceived by the observer must be

$$\nu_O = N/\Delta t_O = \nu_E(\Delta t_E/\Delta t_O)$$

$$= \nu_E . R(t_E)/R(t_O)$$

This gives an expression for the Doppler Effect in the model Universe, which can be written

(2.3) $(1+z) = \nu_E/\nu_O = R(t_O)/R(t_E)$

where z is the usual red shift parameter of observational astronomy. The red shift is seen to be a measure of the relative expansion of the Universe between the time of reception of a photon by the observer and the time of its emission by its source, so that

$$z = \left[R(t_O)/R(t_E)\right] - 1 = \Delta R(t_O, t_E)/R(t_E)$$

For times of emission and reception of the radiation that are sufficiently close, i.e. for photons received from 'local' sources, we can write

$$R(t_E) = R(t_O) - \dot{R}(t_O)\Delta t + \ddot{R}(t_O)\Delta t^2/2 + O(\Delta t^3)$$

where $\Delta t = (t_O - t_E)$ is a small time interval whose higher powers may be neglected. Equation (2.3) could then be written

$$(1+z)^{-1} = R(t_E)/R(t_O) = 1 - f(t_O)\Delta t - \tfrac{1}{2}q_O f^2(t_O)\Delta t^2 + O(\Delta t^3)$$

where q_O is the value of the deceleration parameter (Section 1.3.5) at time t_O and $f(t)$ is the scalar function introduced in Section 1.2.2, Equation (1.1). Inverting this expansion gives

$$1 + z = 1 + f(t_O)\Delta t + f^2(t_O)\Delta t^2(1 + q_O/2) + O(\Delta t^3)$$

so that in the limit $z \ll 1$,

$$(2.4) \quad z = f(t_O)\Delta t + f^2(t_O)(1 + q_O/2)\Delta t^2$$

This is a useful expansion when making 'local' approximations, e.g. when deriving the 'Hubble relations' as in Section 2.5 below.

2.2 Photometric Distances

Photometric distances are the distances that we estimate for objects by assuming them to be 'standard candles', i.e. objects of standard (known) luminosity. If we measure the apparent brightness B (energy falling on unit area of a detector in unit time in a specified instrumental bandwidth) of an object of standard luminosity L, we infer a photometric distance D from

$$(2.5) \quad B = L/4\pi D^2$$

i.e. D is the distance estimated from measured B and assumed L on the assumption of an inverse square law of light propagation and Euclidean geometry. To relate measurable photometric distances to the parameters of model Universes, we need to consider several cases of practical interest.

2.2.1 Bolometric photometry

Consider first the idealised case in which the observer's detector is presumed to respond equally to all wavelengths in the electromagnetic spectrum - it is an ideal bolometer. Consider a source with a bolometric luminosity L, at coordinate σ_E in the model Universe; it emits radiation at time t_E which is observed at the origin $\sigma=0$ at time t_O. Three factors now enter into the computation of the apparent brightness B received by the observer:

1. The area dilution factor. At the time of observation, the radiation from the source is distributed uniformly over a sphere, centred on the source, whose proper area can be found by integrating over all θ and ϕ using the Robertson-Walker metric (~~1.9~~ 1.10) - this area is $A = 4\pi.R^2(t_O).\sigma_E^2$, and the power density at the observer due to purely geometrical dilution is L/A. In addition we must consider two different effects of the red shift:

2. The photon energy loss factor. Every photon emitted by the source with energy $E = h\nu_E$ is received by the observer with energy $E' = h\nu_O$, so the total energy received by the observer is less than that emitted by the source by a factor $f_e = \nu_O/\nu_E = (1+z)^{-1}$.

3. The photon rate factor. N photons emitted in a time interval Δt_E will be received over a longer time interval $\Delta t_O = \Delta t_E(1+z)$, so the rate of photon reception is decreased by a factor $f_r = \Delta t_E/\Delta t_O = (1+z)^{-1}$. Note that this rate factor multiplies the photon energy loss factor f_e. The rate of energy reception from the source per unit area of a bolometric detector is therefore:

$$B(t_O) = f_e.f_r.L/4\pi\sigma_E^2R^2(t_O)$$

$$= L/(4\pi\sigma_E^2R^2(t_O)(1+z)^2)$$

We can therefore write the bolometric distance D_b in terms of σ_E and $R(t_O)$:

$$(2.6)\quad D_b(t_O) = \sigma_E R(t_O)(1+z)$$

To make this an explicit D-z relationship, we need to put in the details of a particular model; before doing this, consider other, more practical, forms for the photometric distance.

2.2.2 Finite-bandwidth photometry

In practice we will use a radiation detector with a band-limited instrumental response $I(\nu_O)$, which will be as constant as possible between frequencies ν_{O1} and ν_{O2}, and as nearly zero as possible at $\nu_O < \nu_{O1}$ and $\nu_O > \nu_{O2}$. In this case the area dilution factor and the photon rate factor f_r enter into the formulation as before, but the photon energy factor

f_e is replaced by the effect of the redshift in moving different parts of the source's radiation spectrum through the instrumental bandpass. For a nearby (zero-red-shift) source the total luminosity in the instrumental bandpass would be

$$L = \int_{\nu_{01}}^{\nu_{02}} I(\nu_0)P(\nu_0)d\nu_0$$

where $P(\nu)$ is the spectral power emitted by the source (luminosity per unit frequency width). For a source at a redshift z however, the radiation received at frequency ν_0 was emitted at frequency $\nu_0(1+z)$, so the effective luminosity of the source in the detector's bandpass is

$$L' = \int_{\nu_{01}}^{\nu_{02}} I(\nu_0)P(\nu_0(1+z))d\nu_0 = L.K_{I,P}(z).(1+z)^{-1}$$

Here the correction factor $K_{I,P}(z)$ depends on the details of the source spectrum and on the shape of the instrumental response curve $I(\nu_0)$, as well as on the red shift z of the source. It may be greater than, or less than, unity, depending on the distribution of energy in the source spectrum $P(\nu)$. These 'K-corrections' therefore have to be determined for particular instruments and particular source spectra, e.g. for the standard photometric U, B, V filters and for elliptical galaxy colours. When the appropriate K-corrections are known, distances D_f estimated photometrically through finite-bandwidth instruments can be converted to equivalent bolometric distances D_b via

(2.7) $$D_b = D_f.\sqrt{K_{I,P}(z)}$$

The determination of the K-corrections is particularly difficult for distant (high-red-shift) objects, as computation of the K-correction in the visible region of the spectrum may require knowledge of the ultraviolet spectra of nearby objects of the same type. Until the advent of satellite observatories, such K-corrections could not be based on observation, but were estimated from theoretical models.

For radio sources, the situation is often simpler. Most radio measurements are made with instruments whose bandwidths are narrow compared with the observing frequency (in order to escape the effects of man-made signals in the radio spectrum), and many radio galaxies and quasars have radio spectra which can be approximated over a wide frequency range by a power law

$$P(\nu) = P_0\nu^{-\alpha}$$

where α is called the spectral index. In this case we can approximate $I(\nu_0)$ with the Dirac delta-function at ν_0 and put

$$K_{I,P}(z) = (1+z).\nu_E^{-\alpha}/\nu_O^{-\alpha} = (1+z)^{1-\alpha}$$

2.3 Diametric Distances

Diametric distances are the distances that we estimate for objects by assuming them to be 'rigid rods', i.e. objects of standard (known) linear size. If we measure the angular diameter $\Delta\theta$ (radians) of an object of known linear size s transverse to the line of sight, we infer a diametric distance d from

(2.8) $\Delta\theta = s/d$

The angle $\Delta\theta$ measures the angle between the paths of two photons which travelled radially towards the observer from the two extreme ends of the source, both setting out at the same time of emission t_E. From the metric (1.9) we see that the element of proper length perpendicular to the radial direction is

$$d\ell^2 = R^2(t)\sigma^2(d\theta^2 + \sin^2\theta d\phi^2)$$

We can eliminate the term involving $\sin^2\theta$ by choosing our angular coordinates such that the source is at $\theta=0$. It follows that the angle $\Delta\theta$ between two radially-travelling photons leaving the ends of an object of finite length s transverse to the radial direction at the same time t_E is

$$\Delta\theta = s/\sigma_E R(t_E)$$

As this angle between the radial photons is preserved as they propagate through the model Universe, we infer that the diametric distance

= proper distance at $t=t_E$ in Euclidean ($k=0$) space

(2.9) $d(t_O) = \sigma_E R(t_E)$

2.3.1 The relation between bolometric and diametric distances

Comparison of our results (2.6) and (2.9) shows that the ratio D_b/d of the bolometric and diametric distances for the same object is

(2.10) $D_b/d = \sigma_E R(t_O)(1+z)/\sigma_E R(t_E) = (1+z)^2$

regardless of the functional form of R(t). There is therefore no information about the cosmological model itself (i.e., about R(t)) to be obtained by comparing bolometric and diametric distances for a given class of object as functions of red shift z. Note also that the ratio of D_b/d becomes large for objects in the presently-known range of red shifts for quasars (z up to 3.5), so that the differences between photometric and diametric distance estimates cannot be ignored for work on the more distant systems accessible to modern telescopes.

2.4 Surface Brightness Relations

2.4.1 Bolometric surface brightness

The surface brightness of an image is its apparent brightness per unit solid angle, i.e. the surface brightness $F(\nu)$ at a given observing frequency ν_0 is

(2.11) $F(\nu_0) = B(\nu_0)/\Omega$

where Ω is the solid angle subtended at the observer by the source. The bolometric surface brightness is $F = \int F(\nu)d\nu$, which for sources whose angular size is independent of frequency is $\int B(\nu)\,d\nu/\Omega$. For a black-body source, the bolometric surface brightness F is proportional to T^4 where T is the effective temperature of the black-body image. For a circular source of proper diameter s we have

$\Omega = \pi s^2/4\pi d^2$ and $B = L/4\pi D_b^2$

so that

$F = (L/\pi s^2).(d/D_b)^2$, i.e.

(2.12) $F \propto (1+z)^{-4}$

which for a black-body image implies that the effective temperature

(2.13) $T \propto (1+z)^{-1}$

Note that this bolometric black-body temperature relation could be written

(2.14) $T(t_O)/T(t_E) = R(t_E)/R(t_O)$

2.4.2 The spectrum of a distant black-body source

Equation (2.13) shows that the temperature T estimated bolometrically from a black-body image falls off as $(1+z)^{-1}$. It is interesting to examine the detailed behaviour of the spectrum of a black body in the model Universe. To do this, imagine that we have a strictly monochromatic detector, so that $I(\nu_O)$ is the Dirac delta function at ν_O in the notation of Section 2.2.2. In that case the K-correction is just

$$K_P(z) = (1+z)P(\nu_O(1+z))/P(\nu_O) = (1+z)P(\nu_E)/P(\nu_O)$$

so that the apparent brightness at frequency ν_O in the observed spectrum is

$$B(\nu_O) = P(\nu_E)/4\pi\sigma_E^2R^2(t_O)(1+z)$$

from Section 2.2.2. We can therefore write the observed surface brightness at frequency ν_O

$$F(\nu_O) = B(\nu_O)/\Omega$$

$$= \frac{P(\nu_E) \qquad .4\pi\sigma_E^2R^2(t_E)}{4\pi\sigma_E^2R^2(t_O)(1+z).\pi s^2}$$

in the notation of Section 2.3. This simplifies to

$$(2.15) \quad F(\nu_O) = F(\nu_E).(1+z)^{-3}$$

Now for a black-body source at a temperature T_E, $F(\nu_E)$ is the emitted Planck spectrum

$$F(\nu_E) = 2h\nu_E^3/c^2\{\exp(h\nu_E/kT_E)-1\}$$

so that the observed surface brightness spectrum is, by (2.14)

$$F(\nu_O) = 2h[\nu_E/(1+z)]^3/c^2\{\exp(h\nu_E/kT_E)-1\}$$

$$= 2h\nu_O^3/c^2\{\exp(h\nu_O(1+z)/kT_E)-1\}$$

This is exactly the spectrum in the observer's frame of the Planck curve at the modified temperature

$$T_O = T_E.(1+z)^{-1}$$

This shows that the effects of the expansion of the Universe on a black-body spectrum are to transform it in frequency so that it retains its Planckian character for all observers, but also so that the observed shape corresponds to different temperatures for observers for whom the source has different red shifts.

2.5 The Local Hubble Relations : z-D and z-d

We will define the 'Hubble relations' to be the approximations for z<<1 to the exact redshift-distance laws which result from our models and equations (2.6) and (2.9). In the limit $z \ll 1$, $\sigma_E \ll 1$, the function $S_k(\sigma_E)$ in Section 2.1 tends asymptotically to σ_E regardless of the value of the curvature constant k. In this case the relationship between a source's parametric coordinate σ_E and the time of emission of the source's radiation, t_E, is

$$\sigma_E(t_E) = \int_{t_E}^{t_O} c\,dt/R(t)$$

Considering σ_E to be a function of t_E we can expand it as a Taylor series around its value at t_O (which is obviously zero by definition):

$$\sigma_E(t_E) = \sigma_E(t_O) - \dot{\sigma}_E(t_O)\Delta t + \ddot{\sigma}_E(t_O)\Delta t^2/2 - \dddot{\sigma}_E(t_O)\Delta t^3/6 + \ldots$$

$$\sigma_E \longrightarrow 0 + \frac{c\Delta t}{R(t_O)} + \frac{c\Delta t^2\dot{R}(t_O)}{2\,R^2(t_O)} + \frac{c\Delta t^3}{6\,R^2(t_O)}\left\{\frac{\dot{R}^2(t_O)}{R(t_O)} - \ddot{R}(t_O)\right\} + \ldots$$

Now use equation (2.6) for the bolometric distance D_b:

$$\begin{aligned} D_b &= \sigma_E R(t_O)\,(1+z) \\ &= c(1+z).\left\{\Delta t + \left[\dot{R}(t_O)/R(t_O)\right]\Delta t^2 + \left[\dot{R}(t_O)/R(t_O)\right]^2(1+q_O)\Delta t^3/6 + \ldots\right\} \\ &= c(1+z).\left\{\Delta t + f(t_O)\Delta t^2 + f^2(t_O)\,(1+q_O)\,\Delta t^3/6 + \ldots\right\} \end{aligned}$$

and substitute for Δt in terms of z using the inverse of expansion (2.4):

$$\Delta t = (z/f(t_O)).\left\{1 - (1+q_O/2)\,z + O(z^2)\right\}$$

so that we can collect together the series expansion

$$D_b = cz/f(t_O).\left\{1 + z(1-q_O)/2 + O(z^2)\right\}$$

which inverts to give

(2.16) $cz = f(t_0)D_b - f^2(t_0)D_b^2(1-q_0)/2c + O(D_b^3)$

We see from this that in the limit of $D_b \to 0$, $z \to 0$, $cz \to v$ (the recessional velocity of the source), the function $f(t_0)$ becomes what we normally term the <u>Hubble constant</u> H_0, i.e. the coefficient of the first-order term in the v-D_b relation. With this identification, we see that

(2.17) $H_0 = \dot{R}(t_0)/R(t_0)$

which will be a function of t_0 in any of our models. Thus the 'Hubble constant' would more properly be termed the 'Hubble parameter'. We will replace the clumsy notation $f(t_0)$ with H_0 in what follows.

Equation (2.9) for the diametric distance may be similarly expanded as a power law in the 'local' approximation to give

$$d = (cz/H_0) \cdot \left\{ 1 - z(3+q_0)/2 + O(z^3) \right\}$$

which inverts to give

(2.18) $cz = H_0 d + H_0^2 d^2(3+q_0)/2c + O(d^3)$

Equations (2.16), (2.17) and (2.18) form the basis for an observational program which could help us decide which, if any, of the theoretical world-models corresponds to the actual Universe around us. By determining the exact velocity-distance laws for standard objects participating in the local Hubble flow we could find the coefficients H_0, A and B in

$cz = H_0 D_b + AD_b^2$ and

$cz = H_0 d + Bd^2$

Obtaining the value of H_0 from the first-order coefficients of these laws, we could then extract q_0 from

$A = -H_0^2(1-q_0)/2c$ and $B = H_0^2(3+q_0)/2c$

Note that this determination of the coefficients in the local Hubble relations would tell us the value of q_0 regardless of whether or not the cosmological constant $\Lambda = 0$. If we are prepared to <u>assume</u> that $\Lambda = 0$, the value of q_0 uniquely fixes the best-fit world-model (see Section 1.3.5). It would be better however to test for the value of Λ via another set of observations.

2.6 The Relation between q_0, ϱ_0 and Λ

If we were successful in determining the value of q_0, we could test for the value of Λ by checking whether this observed value of the deceleration parameter is in fact compatible with pure gravitational deceleration produced by the mean density ϱ_0 of the Universe that exists around us at our observing time t_0. To formulate this test, we return to equation ~~(1.7)~~ 1.8 and make the scaling time t_0 the time t_0 of our observations:

$$3\ddot{R}R^2 = \Lambda R^3 - 4\pi G\varrho_0 R^3(t_0)$$

Now let all quantities take their values at the time $t=t_0$ when we observe the Universe:

$$3\ddot{R}(t_0)R^2(t_0) = \Lambda R^3(t_0) - 4\pi G\varrho_0 R^3(t_0)$$

which can be rearranged to give

$$\Lambda = 4\pi G\varrho_0 + 3.\left[\ddot{R}(t_0)/R(t_0)\right]$$

On substituting for q_0 from ~~(1.12)~~ 1.14 and H_0 from (2.17) we find

$$(2.19)\qquad \Lambda = 4\pi G\varrho_0 - 3q_0H_0^2$$

Observational cosmology - i.e., the search for which, if any, of the solutions to the Friedmann-Lemaître equation corresponds to the mean properties of our Universe - can therefore be regarded as a search for three numbers : H_0, q_0 and ϱ_0, from which we could infer Λ, $R(t)$ and k.

Note that if $\Lambda = 0$, the density

$$(2.20)\qquad \varrho_c = 3H_0^2/8\pi G$$

is a 'critical value' for ϱ_0. If $\varrho_0>\varrho_c$, then $q_0>0.5$ and the $\Lambda=0$ Universe is 'closed'; if $\varrho_0<\varrho_c$, then $q_0<0.5$ and the $\Lambda=0$ Universe is 'open'.

2.7 Exact Distance-Redshift Laws

If we let the quantities in equation ~~(1.6)~~ 1.7 also take their values at $t = t_0 = t_0$, we obtain

(2.21) $kc^2 = R^2(t_0) . \left\{4\pi G\rho_0 - q_0 H_0^2\right\} - \dot{R}^2(t_0)$

Using this and relation (2.19) to eliminate Λ and k from the Friedmann-Lemaître equation ~~(1.8)~~ 1.9 it can be shown that this equation reduces to the form:

(2.22) $dY/dX = 1/Z(Y)$

where $Z^2(Y) = Y/\left[2\sigma_0 + (q_0+1-3\sigma_0)Y - (q_0 - \sigma_0)Y^3\right]$

$Y = R(t)/R(t_0) = (1+z)^{-1}$

$X = H_0 . t$ and $\sigma_0 = 4\pi G\rho_0/3H_0^2$

N=18

This alternate formulation (2.22) of the Friedmann-Lemaître equation is convenient for parameterising its solutions in terms of q_0, H_0 and σ_0.

To obtain exact distance-red shift laws from equations (2.6) and (2.9) we need a prescription for eliminating σ_E from these expressions in favour of the observable, z. The transformed Friedmann-Lemaître equation allows us to show that this can be done in a general case, as follows. We return to (2.1) and its integral

$$S_K(\sigma_E) = \int_{t_E}^{t_0} c\,dt/R(t)$$

Using (2.22) we can transform the right-hand side as follows:

$$\int_{t_E}^{t_0} \frac{c\,dt}{R(t)} = \frac{c}{H_0 R(t_0)}\int_{X_E}^{X_0} \frac{dX}{Y} = \frac{c}{H_0 R(t_0)}\int_{Y_E}^{1} \frac{Z(Y)\,dY}{Y} = \frac{c}{H_0 R(t_0)}\int_{0}^{z} \frac{Z'(z)\,dz}{(1+z)}$$

i.e. $S_K(\sigma_E) = [c/H_0 R(t_0)] . I(z)$, where $I(z)$ is a calculable function of z. As $R(t_0)$ is an arbitrary scaling factor, it will always cancel out from the final result for $D_b(z)$ or $d(z)$. Some special cases have particularly simple solutions for $I(z)$ and thus for the distance-red shift laws.

If $\Lambda = 0$, the general relationship reduces to (W.Mattig, Astr. Nach., 284, 109 (1958)):

(2.23) $D_b = [c/H_0 q_0^2] . \left\{q_0 z + (q_0-1)\left[\sqrt{(1+2q_0 z)} - 1\right]\right\}$

For the Einstein-de Sitter model in particular it reduces to

(2.24) $D_b = (2c/H_0) . \left\{(1+z) - \sqrt{(1+z)}\right\}$

2.8 Source Counts

An observational test that is sometimes attempted in order to discriminate among cosmological models is to make counts of the number N of a given class of source brighter than some limiting brightness B in a standard area of sky. Suppose that the objects have a mean luminosity <L> and that, for simplicity, all quantities are measured bolometrically. Then in terms of the parametric coordinate σ the proper volume dV at coordinate depths between σ_E and $\sigma_E + d\sigma_E$ is, from the Robertson-Walker metric (1.10)

$$dV = R(t_E)d\sigma_E/\sqrt{(1-k\sigma_E^2)} \;.\; 4\pi R^2(t_E)\sigma_E^2$$

$$= 4\pi R^3(t_E).\sigma_E^2.d\sigma_E/\sqrt{(1-k\sigma_E^2)}$$

The number dN of sources in this proper volume is

$$dN = n(t_E).dV$$

$$= 4\pi R^3(t_E)n(t_E).\left[\sigma_E^2/\sqrt{(1-k\sigma_E^2)}\right].d\sigma_E$$

where t_E is the time of emission of the radiation that is received at t_O from sources at this σ_E, and $n(t_E)$ is the proper volume number density of the sources of that class at time t_E. In source-conserving models with no superimposed astrophysical evolution, $n(t_E) = n(t_O)R^3(t_O)/R^3(t_E)$, and $n(t_O)$ is asymptotically the local number density inferred from observations of local volumes of the Universe using either photometric or diametric distance scaling.

From (2.1) we can substitute

$c\,dt_E$ for $R(t_E).d\sigma_E/\sqrt{(1-k\sigma_E^2)}$, so that

$$dN = 4\pi R^2(t_E)n(t_E)\sigma_E^2.cdt_E$$

and the total number of sources N observed back to a given time of emission t* will be

$$N(t^*) = 4\pi cn(t_O)R^3(t_O)\int_{t^*}^{t_O}\left[\sigma_E^2(t_E).dt_E/R(t_E)\right] \tag{2.28}$$

For a given model, with known $R(t)$, σ_E will be a known function of t_E, so this integral will give an explicit form for $N(t^*)$. All that remains in order to derive the associated <u>source count</u> N(B) formula is to convert from t* to B using

$B = \langle L \rangle / 4\pi R^2(t_0)(1+z^*)^2 \sigma^{*2}$

For counts of optical objects, whose spectra allow us to obtain source-by-source red shifts, the last step may not be necessary, as the N(z) relation can (in principle) be observed directly, without using apparent brightness B as a distance indicator. For comparison with such data, (2.28) can be converted to an N(z*) relation.

2.9 The ages of $\Lambda = 0$ models

If we are prepared to assume that $\Lambda = 0$ then analytic relations exist between the age of the model since the (last) singularity at which R(t)=0, and the observed values of the Hubble parameter H_0 and the deceleration parameter q_0. Astrophysical arguments that lead to estimates of the age of the Universe can therefore be used to limit the (H_0, q_0) parameter pair if $\Lambda = 0$. To obtain the age relationships, rewrite equation (1.7)

$$\dot{R}^2 = 2GM_0/R - kc^2$$

in terms of a new variable u defined by

(2.29) $R = (2GM_0/c^2).u^2$

to obtain $\dot{R}^2 = (c/u)^2.(1-ku^2)$, i.e.

$$\dot{R} = (c/u).\sqrt{(1-ku^2)}$$

From the definition of u (2.29) we also have

$$\dot{R} = 2u.\dot{u}.(2GM_0/c^2)$$

and equating these two expressions for $\dot{R}$ gives

$$\dot{u} = \sqrt{(1-ku^2)}.u^{-2}.(c^3/4GM_0)$$

The age of the model at the time of an observation, t_0, is given by

$$t_0 = \int_0^{t_0} dt = (4GM_0/c^3).\int_0^{u_0} u^2/\sqrt{(1-ku^2)}.du$$

where u_0 is the value of u at the time of observation t_0. We can therefore write

(2.31) $t_0 = (2GM_0/c^3).F_k(u_0)$, where

$$F_k(u) = 2\int_0^u u^2/\sqrt{(1-ku^2)}\,.du$$

$$= \sin^{-1}u - u\sqrt{(1-u^2)} \qquad (k = +1)$$

$$= u^3/3 \qquad (k = 0)$$

$$= u\sqrt{(1+u^2)} + \sinh^{-1}u \qquad (k = -1)$$

Equation (2.31) is more useful if we eliminate the scaling mass M_0 by noting that, from (2.17),

$H_0 = \dot{R}(t_0)/R(t_0) = 2\dot{u}_0/u_0 = \sqrt{(1-ku_0^2)}\,.u_0^{-3}.(c^3/2GM_0)$, so that

(2.32) $2GM_0/c^3 = \sqrt{(1-ku_0^2)}\,.u_0^{-3}.\,t_H$

where $t_H = 1/H_0$ is the 'Hubble time' which would be the age of an undecelerated Universe expanding at the observed Hubble rate. From equation (1.14) we can write the deceleration parameter

$q_0 = -R(t_0)\ddot{R}(t_0)/\dot{R}^2(t_0) = 1/2(1-ku_0^2)$, so that

(2.33) $1 - ku_0^2 = 1/2q_0$, and $u_0 = \sqrt{(2q_0-1)/2kq_0}$

Combining (2.31), (2.32) and (2.33) we have

(2.34) $$t_0/t_H = \left(\frac{2kq_0}{2q_0-1}\right)^{3/2}\cdot\left(\frac{1}{2q_0}\right)^{1/2}\cdot\; F_k\left[\sqrt{\frac{2q_0-1}{2kq_0}}\right] \qquad (k \neq 0)$$

For k=0 this is degenerate and the limit follows directly from Section 1.3.2

$$t_0/t_H = 2/3 \qquad (k=0)$$

Knowledge of the astrophysical age t_0 and the Hubble time t_H can therefore be used to estimate q_0 from (2.34), if $\Lambda = 0$.

early t_0-t_H
$\rightarrow$ steady state