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# I N T E R F E R O M E T E R S

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# TRANSFORM THEORY OF RADIO STAR INTERFEROMETERS

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## TRANSFORM THEORY OF RADIO STAR INTERFEROMETERS

In these notes the principles of three common types of radio-star intereferometers in use in radio astronomy will be discussed using the ideas of Fourier transforms in conjunction with the mathematical techniques of convolution. The instruments to be treated are the radio Michelson interferometer, the phase-switched (Ryle) interferometer, and the post-detector (Hanbury Brown and Twiss) interferometer, sometimes referred to as the intensity correlation interferometer. It is thought that the operation of these instruments is so clearly brought out by a discussion in terms of Fourier theory that this approach is greatly preferable to more immediately "physical" arguments.

### 1. BASIC THEORY

#### 1.1 An aperture in coherent illumination

We begin with the Fresnel-Kirchhoff diffraction integral

$$\psi_P = \frac{i}{2\lambda}\iint_S \psi_Q \cdot \left[\frac{e^{ikr}}{r}\right] \cdot \left[\cos(n,r) - \cos(n,r_1)\right] \cdot dS \qquad (1)$$

which relates the disturbance of a scalar wave field $\psi$ at a point P to the distribution of $\psi$ over any surrounding surface, Q being a point on such a surface. Consider at first an aperture in one dimension illuminated by a point source, so that plane wavefronts arrive at the aperture from the object space. We wish to calculate the distribution of signal in the image space, and will use the equation (1) for this purpose. The surface of integration will therefore extend over the entire aperture plane, considered to extend to positive and negative infinities, and also over a hemisphere of such indefinitely large radius that contributions to $\psi_P$ from points on it may be neglected. If the aperture is considered to be in the x dimension, we approximate its effect on the incident plane wave by writing for $\psi_Q$ the "aperture function" defined as $a(x) = 0$ for x lying outside the aperture, and $a(x) = 1$ for x lying inside the aperture. This is equivalent to the first-order perturbation calculation of quantum mechanics, for if the interaction of the incident plane wave with the aperture were included to the full, a(x) would contain terms describing the modification of the incident wave within the aperture. The use of first-order perturbation only (known to optics as St. Venant's Hypothesis) has been found to be adequate for most practical purposes. For the subsequent treatment of radio interferometers it is quite sufficient. We will furthermore observe the variation in signal at points P in the image space whose values of r will all be different. We make the approximation that the phase effect of r (exponential in $r/\lambda$) is very much more than the amplitude effect of r (inverse proportionality). We also make the approximation that we deal in such small angle ranges that the obliquity cosine factor is constant. We can then write

$$\psi_P = \frac{\text{const}}{\lambda}\iint a(x).e^{ikr}.dS$$

$$= \frac{\text{const}}{\lambda}\int_{-\infty}^{+\infty} a(x).e^{ik(R - x.\sin\theta)}.dx$$

where R is the normal distance from the aperture plane to the plane of observation, and $\theta$ is the angle to this normal. The factor $e^{ikR}$ is clearly common to all angles

$\theta$ and can be included in the constant multiplying the whole expression.

Defining the notation

$$X = x/\lambda \qquad S = \sin\theta \tag{2}$$

the integral can be written

$$\psi_P = \text{const.} \int_{-\infty}^{+\infty} a(X).e^{-2\pi iXS}.dX \tag{3}$$

This is the basic result from which all the subsequent treatment will proceed. It tells us that the distribution of signal in angle due to an aperture illuminated by a plane wave (point source response of the aperture) is proportional to the Fourier transform of the aperture function $a(x)$, i.e. to the function $A(S)$ in the conventional notation for Fourier transforms. For a two-dimensional aperture, the analysis can be carried through in an exactly analogous manner to give

$$\begin{aligned} \psi(S_1,S_2) &= \text{const.} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a(x_1,x_2).e^{-2\pi i(X_1S_1 + X_2S_2)}.dX_1.dX_2 \\ &= \text{const.}\ A(S_1,S_2) \end{aligned} \tag{4}$$

If the aperture were supposed to be illuminated by a coherent wave field giving a distribution $f(X)$, then the signal distribution in angle would be given by

$$\psi_P = \text{const.} \int_{-\infty}^{+\infty} a(X).f(X).e^{-2\pi iXS}.dX$$

Now from the convolution theorem for Fourier transforms, we know that the Fourier transform of a product such as $a(x).f(x)$ can be expressed as the convolution of the Fourier transforms $A(S)$, $F(S)$, i.e. as the function $A(S)\circledast F(S)$. Now we have from the general theory of transfer and reception that the output of a system is the convolution of the input with a function representing the effect of the system. Here we see that, if the output is the amplitude distribution in angle of the wave field in the image space, and the input is the amplitude distribution in angle of the wave field in the object space, the transfer function is the Fourier transform of the aperture, represented by the function $a(x)$ or $a(x_1,x_2)$ in the two-dimensional case.

The most general type of aperture function $a(x)$ is a good deal more complicated than that considered on Page 1. It should be written

$$a(x) = a_1(x).e^{ia_2(x)} \tag{5}$$

where the functions $a_1(x)$ and $a_2(x)$ are real functions of x. The function $a_1(x)$ represents the <u>amplitude modulation</u> imposed by the aperture, and the function $a_2(x)$ the <u>phase modulation</u>. The Fourier transform of this most general function can itself be calculated by the use of the convolution theorem. It is :

$$A(S) = \int_{-\infty}^{+\infty} a_1(X).e^{ia_2(X)}.e^{-2\pi XS}.dX$$

This is again the Fourier transform of a pair of functions multiplied together, and the result is therefore the convolution of the individual Fourier transforms. To find the transfer function in amplitude for a general aperture, therefore, we must convolve the Fourier transform of the amplitude modulation function with the Fourier transform of the exponential of i times the phase modulation function.

Before concluding this account of coherent-field illumination, it may be mentioned that the idea of convolution is frequently of use in the calculation of the Fourier transform of the amplitude modulation $a_1(X)$ alone. Suppose the aperture consists of an array of identical slits; the amplitude modulation function $a_1(X)$ is then the convolution of the function $a(X)$ describing an individual slit with an array of $\delta$-functions giving the arrangement of the slits in space. The Fourier transform of $a_1(X)$ is then the product of the Fourier transform of $a(X)$ with the Fourier transform of the $\delta$-function array. The contribution of $a_1(X)$ to the transfer function is frequently easier to calculate by this method than by straightforward integration.

## 1.2 Apertures in incoherent illumination.

In radio astronomy we are often concerned with incoherent wave-fields. The large number of point sources which may be thought of as making up the emitting area of a radio source are not phase-coherent. Although point-source (coherent) theory may be valuable in some applications, the extension of the above treatment to deal with incoherent wave-fields is of greater importance.

Whereas a coherent wave-field is linear in amplitude, so that two different points may contribute disturbances which combine to produce zero intensity at an output, an incoherent wave-field is linear in intensity. Two different points always contribute the superimposed intensity, and interference can only occur between disturbances derived from one point in the wave field. It was shown (eq: 3) that the point-source response in amplitude of an aperture represented by $a(X)$ was the Fourier transform $A(S)$. This being the response to an input which is a $\delta$-function in angle, it is to be regarded as the amplitude transfer function for the aperture. The comparable function in incoherent wave field theory will be the intensity transfer function, which is clearly $A(S).A^*(S)$, where the * denotes the complex conjugate. In the analysis of the radio star interferometers, the analogue of the "aperture function" $a(X)$ will be required, and this will now be derived.

This analogue will be called the intensity transmission function, and will be denoted by $T(X)$. The one-dimensional case will be considered for reasons of simplicity in the working. The treatment of the two-dimensional case is strictly similar.

If $A(S).A^*(S)$ is the Fourier transform of $T(X)$, then

$$T(X) = \int_{-\infty}^{+\infty} A(S).A^*(S).e^{2\pi iSX}.dS$$

$$\text{But} \quad A(S) = \int_{-\infty}^{+\infty} a(X).e^{-2\pi iSX}.dX = \int_{-\infty}^{+\infty} a(X').e^{-2\pi iSX'}.dX'$$

$$\text{Thus} \quad T(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a(X').A^*(S).e^{2\pi i(X-X')S}.dS.dX'$$

$$T(X) = \int_{-\infty}^{+\infty} a(X').a^*(X-X').dX'$$

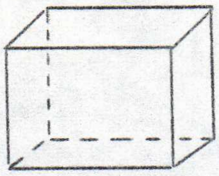
Therefore the intensity transmission function $T(X)$ for an aperture with aperture function $a(X)$ is the self-convolution of $a(X)$. This theorem is closely parallel to the Wiener-Khinchine theorem for stochastic processes. The method of dealing with incoherent wave fields now follows closely that used for coherent wave fields, $T(X)$ replacing $a(X)$, and $A(S).A^*(S)$ replacing $A(S)$.

A number of such self-convolutions will now be presented graphically.

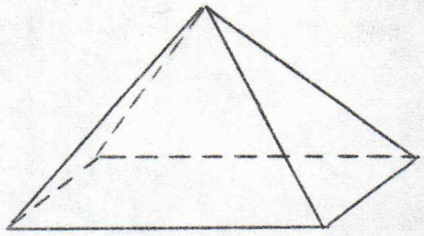
## 1.3 Self-convolutions of some important functions.

a) Rectangular aperture

a(X)



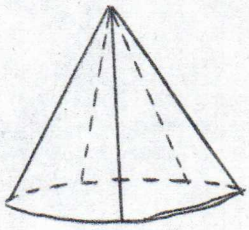
T(X)



b) Circular aperture

a(X)

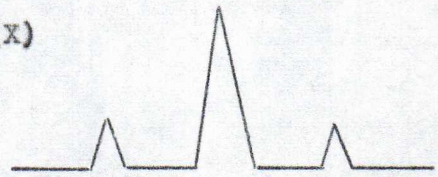
T(X)



c) Two one-dimensional slits, in phase

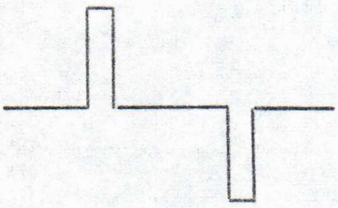
a(X)

T(X)

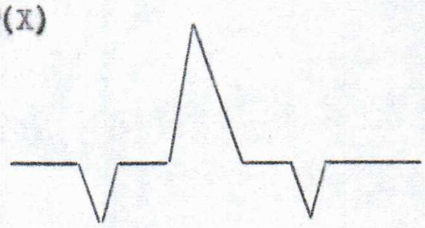


d) Two one-dimensional slits, one $\pi$ out of phase with other

a(X)



T(X)



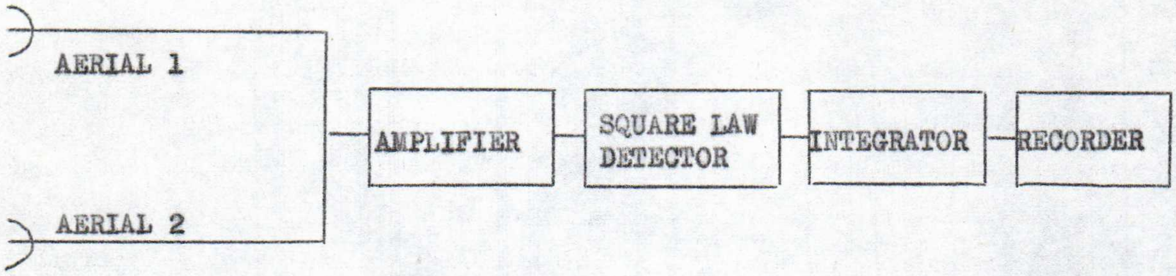
The behaviour of a single aperture in incoherent illumination (1.3a) has the characteristics of a low-pass filter, the response to angular frequencies in the incident disturbance being greatest for $S = 0$. (It should be noted that if we illuminate an aperture with a source having an intensity distribution $p(X_s)$, the wave field in the aperture, $\psi(X_a).\psi^*(X_a)$ is proportional to the Fourier transform $P(S)$ when the aperture is sufficiently far away from the source. Thus when we considered above the response of an aperture to a point source having a $\delta$-function for $p(X_s)$, the wave field at the aperture is represented by a function $f(X)$ which is a constant, all angular frequencies being present in the incident disturbance.) In this respect the response of a single aperture to incoherent illumination is similar to the response of a low-pass filter to a random noise signal, the output of the filter being taken through a square-law detector.

We shall now apply these theoretical ideas to the various radio interferometers mentioned at the outset, beginnning with the Michelson instrument, named after the originator of the corresponding instrument in visual optics.

## 2. THE RADIO MICHELSON (STELLAR) INTERFEROMETER

### 2.1 Block diagram of the system

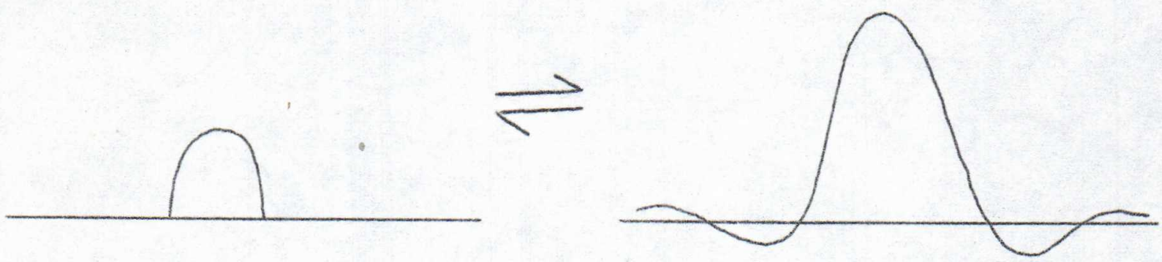
The radio analogue of the optical Michelson stellar interferometer consists of two small aerials separated by a large number of wavelengths at the observing frequency and connected to a common input through lines of equal path length. The combined input is amplified, square-law detected and fed to an integrator and recorder. The arrangement is represented in block form below :




Radio Michelson (Stellar) Interferometer

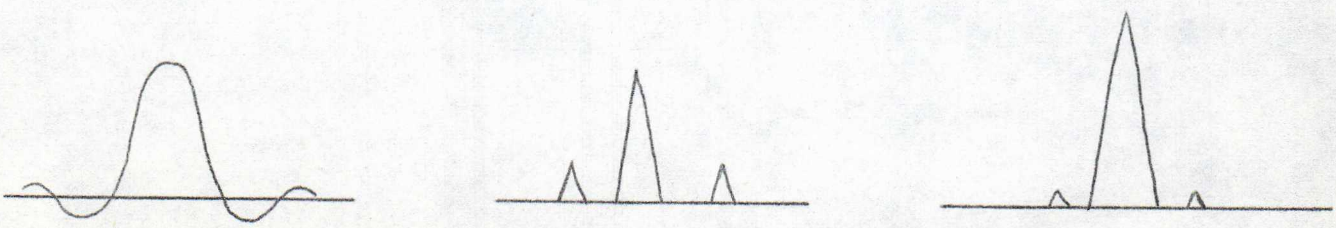
### 2.2 Transform analysis of the system

The distribution of radio intensity across the distant star will have the form of the function in diagram a) below. As the interferometer is situated in effect infinitely far away from the star, the distribution of intensity with separation of the aerials on the Earth's surface is the distribution in angle at the star, i.e. the Fourier transform of the function of diagram a). This is represented by diagram b).



a) b)

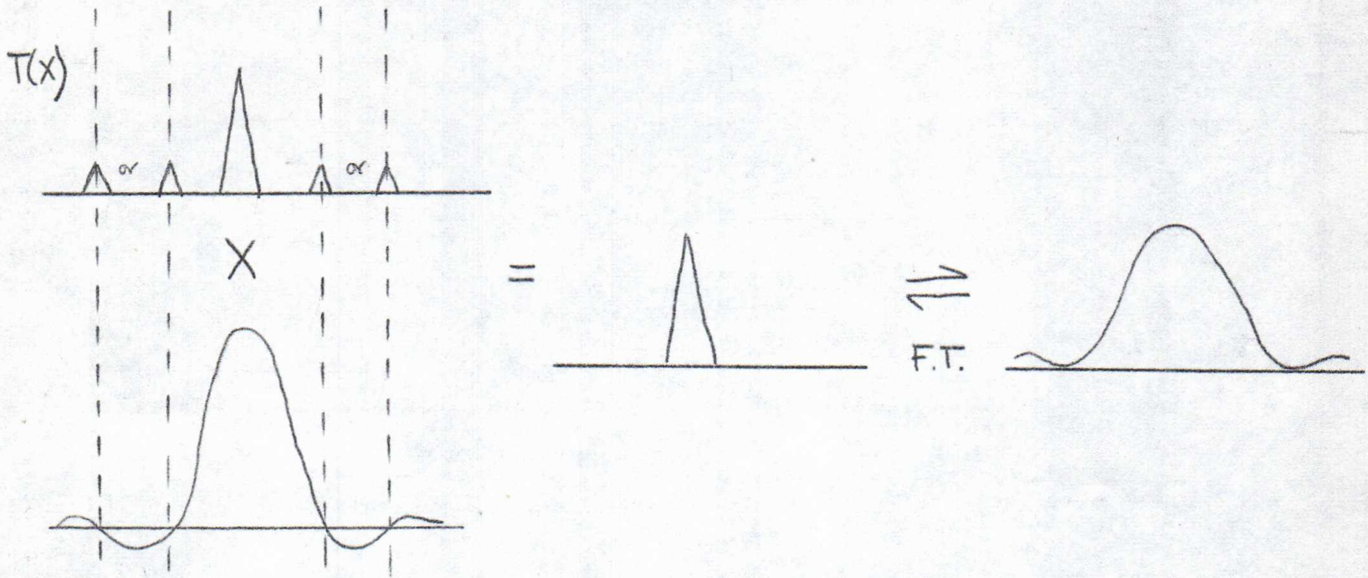
To determine the output of the receiver, i.e. the form of the tracing of a pen recorder as the star passes through the aerial beam due to the rotation of the Earth, we merely multiply the function in b) by the intensity transmission function of the aerials. This is the self-convolution of the aperture function, from section 1.2. If we consider the idealised case where the aerials have a primary reception pattern which is constant over a small range of angles, but then falls immediately to zero outside this range, the aperture function is the a(X) of 1.3 c), and the intensity transmission function the T(X) of the same diagram. The output of the receiver is the Fourier transform of the product of this T(X) with the distribution b), as drawn below :



DISTRIBUTION WITH SPACING X TRANSFER FUNCTION =

PRODUCT F.T. OUTPUT

It is seen that the output of the recorder consists of a D.C. level determined by the Fourier transform (in intensity) of one of the aerials (centre component of product function), modulated by a component proportional to the height of the intensity-separation function at the spacing of the side-peaks in the intensity transfer function of the interferometer. If therefore the aerial spacing is adjusted so that these side-peaks correspond with the first minimum of the intensity-separation distribution (i.e. the first zero of the intensity-angle distribution at the star), there is no modulation, and the output is a D.C. level given only by the Fourier transform of the intensity transmission function of one aerial. This case is shown diagrammatically below :



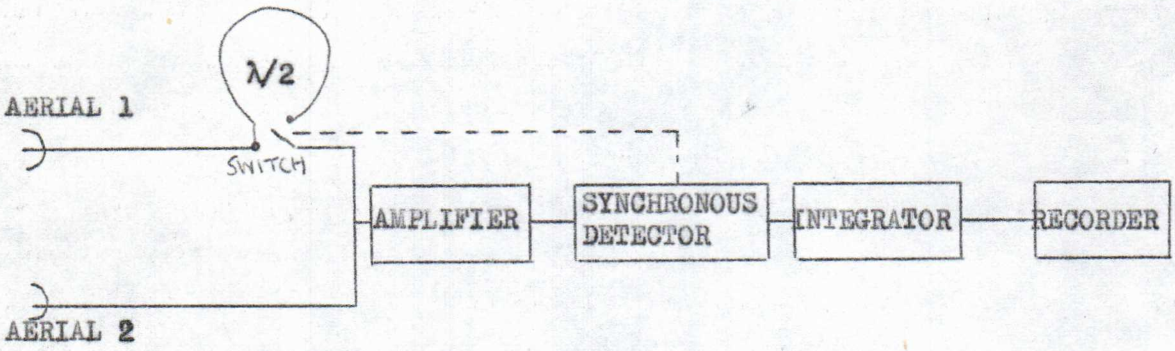

In order therefore to determine the angular diameter of a distant star with this instrument, observations have to be made using a range of aerial spacings until an output is obtained that is simply the unmodulated output of a single aerial. From the spacing of the aerials giving this condition the first zero of the intensity-angle distribution at the star is known, and hence the angular diameter. It will be noticed that the interferometer effectively determines the magnitude of the Fourier component of the intensity-separation distribution at the Earth's surface corresponding to a spatial periodicity equal to the aerial spacing.

## 3. The PHASE-SWITCHED (RYLE) INTERFEROMETER

### 3.1 General discussion and block diagram

The Michelson interferometer described above has the disadvantage that it produces at the output a D.C. level containing no information about the star under investigation, this level being purely a property of the aerial elements which make up the interferometer. The salient feature of the Ryle interferometer is that this level is removed, enabling much greater sensitivities to be obtained on the output recorder, and also discriminating against the large isotropic (roughly) amount of background radiation against which the star will anyway be viewed. This background level is very slowly varying in angle as seen from the Earth, and its significantly large Fourier components are mostly confined to the range let through by the central peak in the intensity transmission function of the Michelson interferometer. Thus the phase-switched system enables us to discriminate effectively between the stars we wish to observe and fluctuations of the (mostly Galactic) background emission. The only background fluctuations confusing the output of the switched instrument are those with Fourier components lying in the side-peaks of the intensity transmission function. These are usually very small components as the background is largely only slowly varying.

The block diagram of the system is given below, and the following discussion will show that it does indeed remove the central peak (spatial frequency pass-band, as it might be termed) from the intensity transmission function.

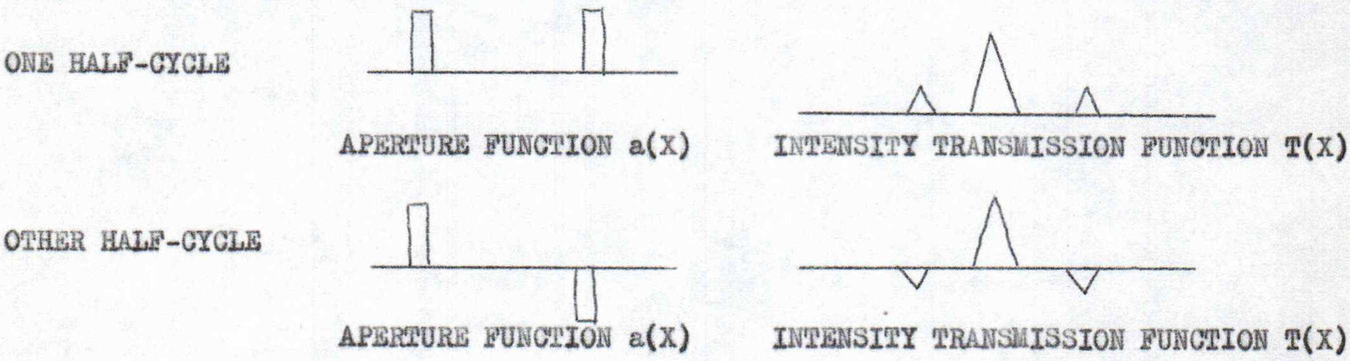



Block diagram of phase-switched stellar interferometer

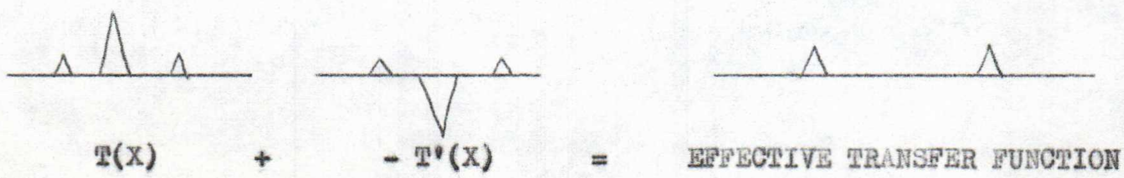
The two aerials are here connected through equal path length cables as before, but an additional $\lambda/2$ of cable is rapidly switched in and out of one path. Synchronously with this operation, the output of the detector is reversed (the detector is a square-law detector as before), the combined operation of detection and synchronous switching being included in the "synchronous detector" of the block diagram above.

## 3.2 Transform analysis of the phase-switched interferometer.

In the half-cycle of the switching waveform during which the extra $\lambda/2$ of path length is not included, the intensity transmission function of the system is the same as that of the Michelson interferometer discussed above. In the other half-cycle, however, it takes the form of diagram 1.3 d), as one aerial is $\pi$ out of phase with the other. Therefore we have




Now, the operation of reversing the detector output synchronously with the switching waveform, and then adding the results by an integration process with a time-constant much longer than the frequency-period of the switch, gives rise to an effective intensity transmission function which is the difference of these two functions, i.e.




It follows from the absence of the central peak in this function that the D.C. component appearing in the output of the Michelson interferometer is absent from the output of this system. The aerial spacing at which the output of the interferometer is reduced to zero now gives the first zero in the Fourier transform of the source intensity distribution.

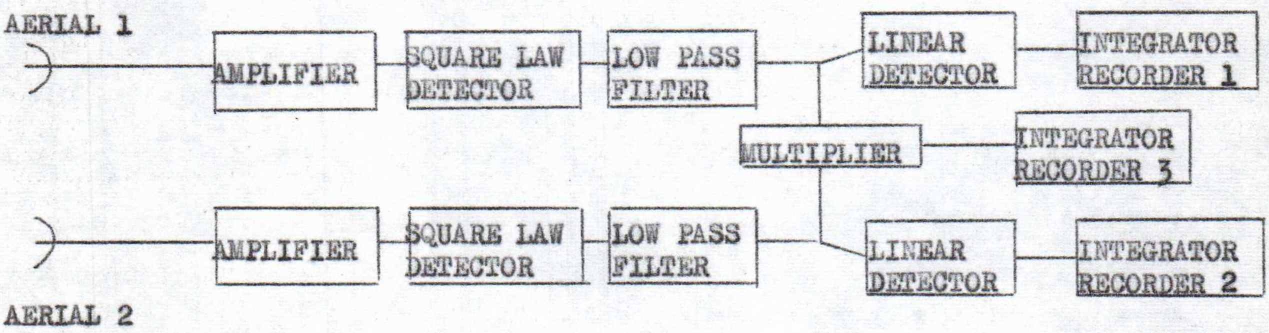
It is no longer possible with this system to build up the Fourier transform of the source intensity by means of different aerial spacings, as the Michelson visibility criterion has no meaning in this system, only the modulation and not the background D.C. level being present. The advantages of the system have been mentioned above, however, and are described in considerable detail by Ryle ("A new radio interferometer and its application to the observation of weak radio stars", Proc. Roy. Soc. A, 211, 351, (1952) )

## 4. THE POST-DETECTOR (HANBURY BROWN and TWISS) INTERFEROMETER

### 4.1 General discussion

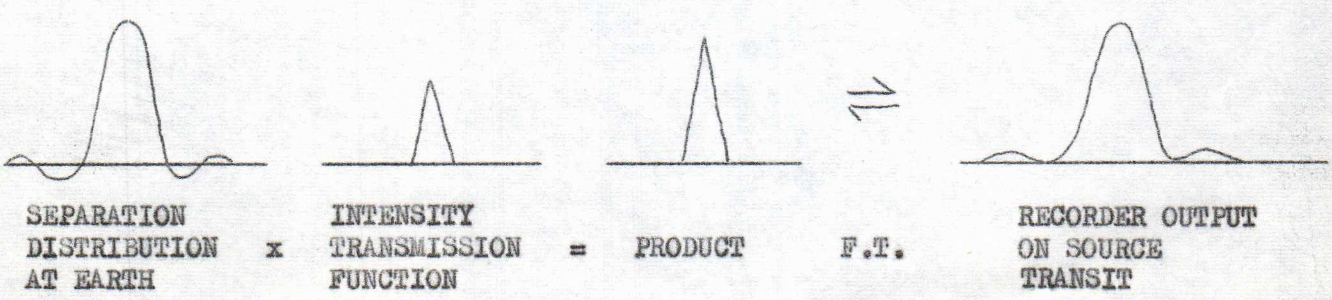
In this interferometer use is made of the result that the correlation coefficient between the disturbances at two points in a wave field is related to the intensity distribution across the source producing the wave-field in a Fourier transform manner (the normalised correlation coefficient equals the normalised Fourier Transform of the intensity distribution in the source, from the Wiener-Khinchine Theorem, see "Correlation, Visibility, and Coherence of Wave Fields", uniform with this discussion.), and the interferometer provides a means of measuring this correlation coefficient fairly directly. The instrument has some important advantages, which are reviewed in the original paper by Hanbury Brown and Twiss (Phil. Mag. 45, 663, 1954). In particular it is free from the effects of scintillations caused by the ionosphere, and connexion between the two aerials can be made by radio link or by later analysis of synchronised recordings.

The two aerials feed separate amplifiers and square-law detectors, which are followed by low-pass filters. The outputs from these filters are in the first instance passed through linear detectors, integrated and recorded, and also multiplied together, integrated and recorded. The block diagram follows :

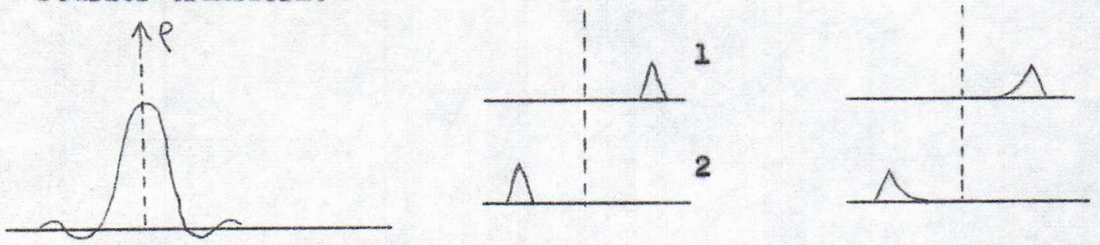



### 4.2 Transform analysis of operation

The output of each of the filters may readily be found, the intensity transfer function of a single aerial being described in 1.3 a). The output of the filters is the same as the output of each of the recorders 1 and 2, and is derived as follows :



Although the use of two separate receivers for the two aerials and subsequent square-law detection has destroyed information about the absolute phases of signals arriving at the two aerials, relative phase information remains, and to analyse the output of the cross-product channel we must therefore deal with a "relative" intensity transmission function for each aerial. This is clearly the intensity transmission function used above, displaced relative to the intensity-separation curve. As this latter is the Fourier transform of the source intensity distribution, its ordinate at any point corresponds to the value of the correlation coefficient between the wave disturbances at sampling points separated by the given spacing. Each of the aerials contributes a term of this kind, and as is shown below, the output of one channel is the reflection about the centre position of the output of the other (note that the separation of the two intensity transmission functions in the diagram corresponds therefore to twice the separation of the aerials in space). The operation of multiplication in the multiplier unit and subsequent integration corresponds to convolving the two channels in the Fourier transform representation, so to obtain the output of recorder 3 we convolve the two product outputs with one another and take the Fourier transform.




INTENSITY-SEPARATION CURVE (CORRELATION COEFFICIENT) X RELATIVE INTENSITY TRANSMISSION = PRODUCT


PRODUCT 1 convolved with PRODUCT 2 = CONVOLUTION F.T. gives OUTPUT 3

The symbol $\rho$ denotes the value of the correlation coefficient between the disturbances at the two aerials for a given separation. This is measured directly by the radio Michelson interferometer discussed in section 2. The square of the normalised correlation coefficient is determined by this system if the output of channel 3 is divided by the square root of the product of the outputs of channels 1 and 2. The correlation coefficient will obviously vanish throughout the entire transit of a source if the aerial spacing takes values such that the relative intensity transmission functions sample the intensity-separation curve at its minima, and so the angular diameter of a source can be inferred. The intensity distribution at the source cannot be found uniquely from a set of correlation coefficients for different aerial spacings because phase information has been lost (this also applies to the Ryle interferometer used as described above). The problem is somewhat similar to that of deducing a crystal structure from its Fourier transform when phases are not known. Assessments of the relative merits of the various systems treated here have been made in detail in the literature of radio astronomy and will not be repeated here.

5. NOTES AND CORRIGENDA