Propagation in Lossy and Superconducting Cylindrical Waveguides

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Abstract – We present rigorous analysis of guided propagation in cylindrical waveguides with finite conductivity and superconducting walls. Our calculations are based on a method by Stratton which solves Maxwell's equations in cylindrical coordinates. The transcendental equation gives the complex propagation constant as a function of frequency, geometry and conductivity. The complex conductivity of the superconducting waveguide is obtained from BCS theory. We computed the attenuation and cutoff frequencies in different materials at microwaves and submillimetre-wave frequencies and compared with those obtained from the commonly used approximate computations.

1 Introduction

While the electromagnetic behaviour of planar superconducting transmission lines have been thoroughly investigated [1] – [4], little has been reported on superconducting waveguides. Waveguides are fundamentally different from the microstrip. A microstrip can support a single TEM with complex propagation constant that remains constant below the superconducting gap. A cylindrical waveguide however supports multimode operation and the complex conductivity influences propagation via surface impedance dependency and also by modifying the modes cuttoff. Consequently we expect superconductivity to influence attenuation significantly, not only near the gap but also near cutoff.

The commonly used method for calculating the complex propagation constant in a waveguide is to first obtain the fields by assuming infinite conductivity. This allows separating the solution into TE and TM modes. The cutoff frequencies and phase velocity are obtained by solving a simple characteristic equation. To calculate attenuation, the fields are assumed to penetrate the conductor surface and energy dissipated rapidly within a thin layer. This analysis is only valid if the decay of the field within the surface is much faster than its variation in the tangential plane, an assumption that applies for good conductors.

In a waveguide of finite conductivity, pure TE or TM modes cannot be excited separately. A valid solution of the wave equation must be expanded in terms of both TE and TM mode functions. Equating the tangential components of the fields at the boundary within and outside the conductor surface yields characteristic equations far more complicated than those for the perfect conductor.

2 Propagation in lossy cylindrical waveguides

Approximate solution: The electric and magnetic fields propagating in the z-direction in a waveguide with uniform cross section are expressed as -

$$\mathbf{E} = \mathbf{E}_0 e^{-(\alpha + j\beta)z} \text{ and } \mathbf{H} = \mathbf{H}_0 e^{-(\alpha + j\beta)z}, \tag{1}$$

where α and β are respectively the attenuation and phase constants. By considering the average power per unit area along the waveguide and using the Poynting theorem, the attenuation constants for the TM and TE modes respectively are given by the approximate expressions [5] -

$$\alpha = Re\left[\frac{Z_S}{a\eta\sqrt{1-(f_c/f)^2}}\right], Re\left\{\frac{Z_S}{a\eta\sqrt{1-(f_c/f)^2}}\left[(f_c/f)^2 + \frac{(n/p'_{nl})^2}{1-(n/p'_{nl})^2}\right]\right\}, \quad (2)$$

where a is the radius, Z_S the surface impedance, η the intrinsic impedance, f the frequency, f_c the cutoff frequency, and p'_{nl} is the l-th root of the first derivative of the Bessel function J'_n.

<u>*Exact solution*</u>: The field in the cylindrical waveguide may be written as a combination of elementary waves having the general functional form –

$$\psi = e^{jn\phi} F_n(k_c r) e^{\pm j\gamma z - j\omega t}, \quad (3)$$

where F_n is a cylindrical function, $\gamma = \alpha + j\beta$ is the propagation constant, $k_c^2 = k^2 - \gamma^2$, $k = \sqrt{\mu\varepsilon\omega^2 - j\mu\omega\sigma}$, and σ is the complex conductivity of the lossy waveguide wall. For superconducting waveguide the complex conductivity is obtained from BCS theory [7]. Within the waveguide ($0 \le r \le a$), F_n takes the form of a Bessel function J_n and outside the guide (dielectric or lossy conductor), the Hankel function of the first kind $H_n^{(1)}$ is used to satisfy the radiation condition at $r \to \infty$. The boundary condition that the tangential fields are continuous across the waveguide wall yields four linear-homogeneous equations in four unknown coefficients. A non-trivial solution is only obtained if the determinant of the equations vanishes. This yields the transcendental equation [7] -

$$\left[\frac{\mu_1}{u}\frac{J'_n(u)}{J_n(u)} - \frac{\mu_2}{v}\frac{H_n^{(1)'}(v)}{H_n^{(1)}(v)}\right]\left[\frac{k_1^2}{\mu_1 u}\frac{J'_n(u)}{J_n(u)} - \frac{k_2^2}{\mu_2 v}\frac{H_n^{(1)'}(v)}{H_n^{(1)}(v)}\right] = n^2\gamma^2\left(\frac{1}{v^2} - \frac{1}{u^2}\right)^2. \quad (4)$$

In the above $u = k_c^1 a$, $v = k_c^2 a$ and the superscripts 1, 2 refers to the regions inside and outside the waveguide respectively. The above equation can be solved numerically for the propagation constant γ for TE modes. For TM modes an alternate form of the equation is required -

$$\left[\frac{J_n(u)}{J'_n(u)}\right]^2\left[\frac{H_n^{(1)'}(v)}{H_n^{(1)}(v)}\right]^2\frac{k_2^2}{v^2} + \frac{k_1^2}{u^2} - \frac{1}{uv}\frac{J_n(u)}{J'_n(u)}\frac{H_n^{(1)'}(v)}{H_n^{(1)}(v)}\left(\frac{\mu_1}{\mu_2}k_2^2 + \frac{\mu_2}{\mu_1}k_1^2\right) = \left[\frac{J_n(u)}{J'_n(u)}\right]^2 n^2\gamma^2\left(\frac{1}{v^2} - \frac{1}{u^2}\right)^2 \quad (5)$$

3 Results & Conclusion

The results are depicted in Fig. 1 – 6. Our analysis reveals that in practical applications, the modes in a superconducting waveguides could be approximated to those in a perfect conductor, which is consistent with recently reported experimental results. We also found that for good conductors, the attenuation computed by the surface impedance method is very close to the rigorous solution. However the results from the two methods differ significantly in two regions. They deviate near cutoff and at extremely high frequencies. At $\omega = \omega_c$ the attenuation given by the approximate method becomes singular. In the

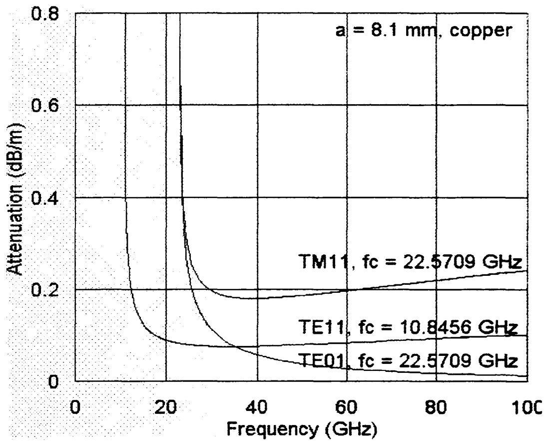

Fig. 1 Attenuation for TE11, TE01 and TM11 modes in a copper waveguide of radius 8.1 mm at low frequencies. Results by both the exact and the approximate surface impedance methods are in very good agreement.

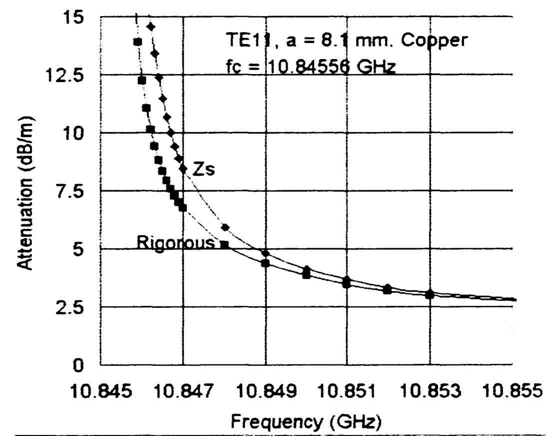

Fig. 2 Attenuation for TE11 mode in a copper waveguide of 8.1 mm as in Fig. 1 showing discrepancy between the results of the rigorous and the surface impedance methods near the cutoff frequency.

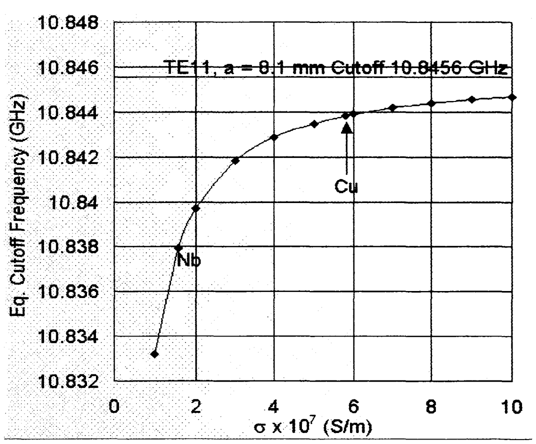

Fig. 3 Cutoff frequency as a function of conductivity. Notice that the cutoff calculated by the approximate method does not vary with conductivity.

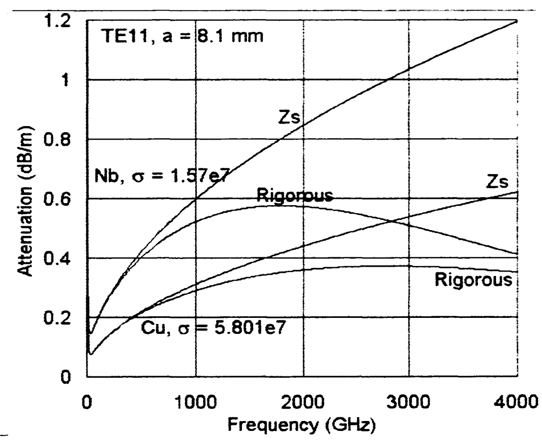

Fig. 4 Attenuation for TE11 mode in a copper waveguide of 8.1 mm as that in Fig. 1 at extremely high frequencies.

exact solution such singularity does not exist and the attenuation diverges sharply but continuously. The differences turn out to be significant even for good conductors. This difference in the attenuation results is large enough to be easily measurable. Next, at very high frequencies, with the approximate method we can still assume that only the TE_{11} mode is excited and hence the attenuation as a function of frequency diverges to infinity. The exact solution however gives a finite loss, which is clearly a more realistic behaviour. We attribute these differences to the fact that in those cases the field can no longer be approximated to those of a perfect conductor. In particular the solutions are no longer separable as either TE or TM modes.

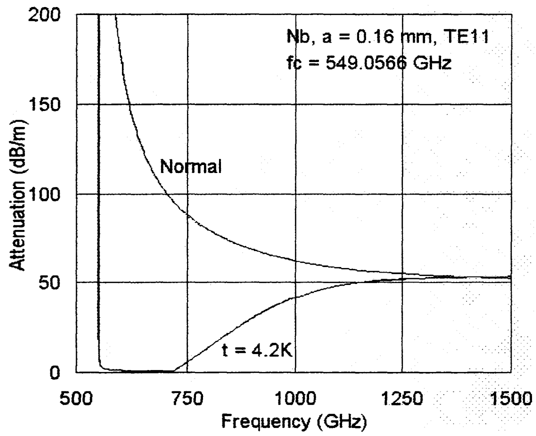

Fig. 5 Attenuation for TE11 mode in a Nb waveguide of 0.16 mm. Both the rigorous and surface impedance methods agree well for both normal and non-superconducting states.

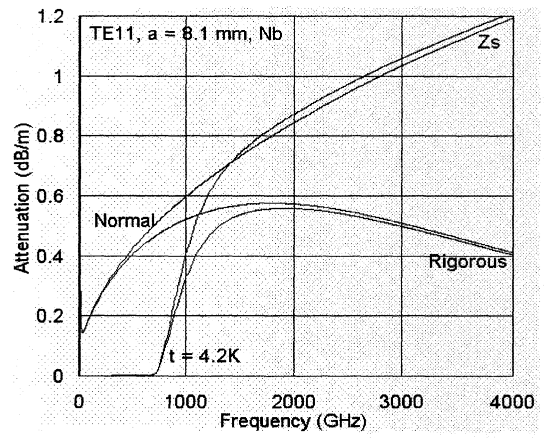

Fig. 6 Attenuation for TE11 mode in a Nb waveguide of 8.1 mm. Both normal and superconducting behaviours are shown. At low frequencies both the rigorous and surface impedance methods agree well. There are deviations between the two methods at high frequencies above the gap.

References

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